

Painlevé equations, topological type property and reconstruction by the topological recursion

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Abstract

In this article, we prove that we can introduce a small \hbar parameter in the six Painlevé equations through their corresponding Lax pairs and Hamiltonian formulations. Moreover, we prove that these \hbar -deformed Lax pairs satisfy the Topological Type property proposed by Bergère, Borot and Eynard for any generic choice of the monodromy parameters. Consequently we show that one can reconstruct the formal \hbar series expansion of the tau-function and of the determinantal formulas by applying the so-called topological recursion on the spectral curve attached to the Lax pair in all six Painlevé cases. Eventually we illustrate the former results with the explicit computations of the first orders of the six tau-functions.

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1 Introduction

It is now well known that the study of hermitian random matrices is intrinsically related to integrable systems and Painlevé equations. In particular it is proved that the partition functions of hermitian matrix models define isomonodromic tau-functions [5, 6]. Moreover, (see for example

[31] for a review) it is also well understood that the local correlations between consecutive eigenvalues in hermitian random matrices exhibit universal behaviors when the size of the matrix goes to infinity. For example, the gap probability in the bulk of the distribution of eigenvalues can be directly connected to the Painlevé V equation, the so-called Hastings McLeod solution of the Painlevé II equation is related to the gap probability at the edge, and many similar results in different locations of the eigenvalues distribution are now available in relation with the other Painlevé equations. In a more algebraic perspective, it was recently realized that the connection between integrable systems and hermitian matrix models can be understood because of the existence of loop equations (also known as Schwinger-Dyson equations) that can be solved perturbatively (under additional assumptions like convex potentials or genus 0 spectral curve) throughout the topological recursion introduced by Eynard and Orantin in [16]. Since the scope of the topological recursion has been proved to go much beyond matrix models, it is natural to wonder if one could define the equivalent of correlation functions arising in the formalism of the topological recursion directly into the integrable systems formalism. In [2] and [3], Bergère, Borot and Eynard suggested determinantal formulas that associate to any Lax pair (in fact any finite dimensional linear differential system) a set of correlation functions that satisfy the same loop equations as the one arising in hermitian matrix models. Consequently at the perturbative (i.e. formal) level, one may expect these correlation functions to be reconstructed by the topological recursion. However, since loop equations may have many solutions it is not obvious why the functions generated by the topological recursion (that are one set of solutions to the loop equations) should necessarily identify with the determinantal formulas (that are also one set of solutions to the loop equations). In [2] and [3] the authors discussed about sufficient conditions on the Lax pair, known as the Topological Type property, on the Lax pair to prove that both sets are identical. The purpose of this article is to show that in the case of the six Painlevé equations with generic monodromy parameters, one can introduce a perturbative formal parameter \hbar in the Lax pair and prove the Topological Type property in all six cases. We introduce a formal \hbar parameter in the Painlevé Lax pairs (historically proposed by Jimbo, Miwa and Ueno in [22] and [23]) through a rescaling of the parameters thus providing a \hbar deformation of the Painlevé equations whose standard forms can be recovered by taking $\hbar = 1$. We also provide \hbar -deformations of the Hamiltonian structures underlying the Painlevé equations [34]. Our results extend similar results developed for the Painlevé II equation in [19] as well as partial results (vanishing monodromies and an incomplete proof via the insertion operator) for the Painlevé V equation in [30]. Our analysis is sufficiently general to cover all six Painlevé cases and provides many interesting remarks that would deserve closer attention. In particular it appears that only specific features of the Lax systems are involved in the computation of the determinantal formulas and that all physically relevant quantities are invariant under reasonable gauge transformations of the Lax pairs.

Our paper is organized as follow: First, we present the six Lax pairs, Hamiltonians and Jimbo-Miwa-Ueno tau-functions describing the six Painlevé equations. Since our gauge choices and notations are slightly different from the historical ones given by Jimbo and Miwa in [23] we also provide the explicit correspondences in appendix A. Then we present the natural rescaling of the parameters thus introducing our formal expansion parameter \hbar . In section 5 we present the computation and analysis of the spectral curves. Section 6 is then dedicated to the presentation of the determinantal formulas and of the Topological Type property. Finally in section 6.3 we get to the statement of our main theorem, i.e. that all six Lax pairs satisfy the Topological Type property. The proof of the Topological Type property is postponed in appendices E, F, G and H and follows the same strategy as the one developed in [19]. Eventually conclusions and outlooks are presented in section 7 and computations of the unstable cases $F^{(0)}$ and $F^{(1)}$ are presented in appendix I.

2 Painlevé equations, Lax pairs, Hamiltonians and τ -functions

2.1 Lax pairs

Painlevé equations play an important role in the theory of integrable systems. They were studied originally by Garnier and Painlevé and their connections with integrable systems was detailed later by Jimbo, Miwa and Ueno in a series of famous papers [22, 23, 24]. In this series of papers, the authors provide various 2×2 Lax pairs from which one can reconstruct the Painlevé equations through the compatibility equations of the systems. Since then, many adaptations and other Lax pairs have been proposed to recover these Painlevé equations. In this paper we will use Lax pairs that can be directly connected to the Jimbo-Miwa pairs and we provide in appendix A the correspondences between our Lax pairs and the Lax pairs proposed by Jimbo and Miwa in [23].

Definition 2.1 (Lax pair) *A Lax pair corresponds to the definition of two $n \times n$ matrices $\mathcal{D}(x; t)$ and $\mathcal{R}(x; t)$ such that the system:*

$$\partial_x \Psi(x, t) = \mathcal{D}(x, t) \Psi(x, t) \quad , \quad \partial_t \Psi(x, t) = \mathcal{R}(x, t) \Psi(x, t)$$

is consistent. In the theory, x is usually called the spatial parameter while t provides the so-called time-evolution of the system. The compatibility equations (also known as zero-curvature equations) are given by:

$$\partial_t \mathcal{D}(x, t) - \partial_x \mathcal{R}(x, t) + [\mathcal{D}(x, t), \mathcal{R}(x, t)] = 0$$

In [23], Jimbo and Miwa provide 2×2 matrices $(\mathcal{D}_J(x, t), \mathcal{R}_J(x, t))$ with $1 \leq J \leq 6$. We propose here adaptations of these Lax pairs:

- For (P_I) :

$$\mathcal{D}_I(x, t) = \begin{pmatrix} -p & x^2 + qx + q^2 + \frac{t}{2} \\ 4(x - q) & p \end{pmatrix} \quad , \quad \mathcal{R}_I(x, t) = \begin{pmatrix} 0 & \frac{x}{2} + q \\ 2 & 0 \end{pmatrix} \quad (2.1)$$

- For (P_{II}) :

$$\mathcal{D}_{II}(x, t) = \begin{pmatrix} x^2 + p + \frac{t}{2} & x - q \\ -2(xp + qp + \theta) & -(x^2 + p + \frac{t}{2}) \end{pmatrix} \quad , \quad \mathcal{R}_{II}(x, t) = \begin{pmatrix} \frac{x+q}{2} & \frac{1}{2} \\ -p & -\frac{x+q}{2} \end{pmatrix} \quad (2.2)$$

- For (P_{III}) :

$$\begin{aligned} \mathcal{D}_{III}(x, t) &= \begin{pmatrix} \frac{t}{2} - \frac{\theta_\infty}{2x} + \frac{p-\frac{t}{2}}{x^2} & -\frac{pq}{x} - \frac{p}{x^2} \\ \frac{-(p-t)q - \theta_\infty + \frac{t(\theta_0 + \theta_\infty)}{2p}}{x} + \frac{p-t}{x^2} & -\left(\frac{t}{2} - \frac{\theta_\infty}{2x} + \frac{p-\frac{t}{2}}{x^2}\right) \end{pmatrix} \\ \mathcal{R}_{III}(x, t) &= \begin{pmatrix} \frac{x}{2} - \frac{p-\frac{t}{2}}{tx} + \frac{\theta_0 + \theta_\infty}{2p} + q - \frac{\theta_\infty}{2t} & -\frac{pq}{t} + \frac{p}{tx} \\ \frac{-(p-t)q - \theta_\infty + \frac{t(\theta_0 + \theta_\infty)}{2p}}{t} - \frac{p-t}{tx} & -\left(\frac{x}{2} - \frac{p-\frac{t}{2}}{tx} + \frac{\theta_0 + \theta_\infty}{2p} + q - \frac{\theta_\infty}{2t}\right) \end{pmatrix} \end{aligned} \quad (2.3)$$

- For (P_{IV}) :

$$\begin{aligned}\mathcal{D}_{IV}(x, t) &= \begin{pmatrix} x + t + \frac{pq + \theta_0}{x} & 1 - \frac{q}{x} \\ -2(pq + \theta_0 + \theta_\infty) + \frac{p(pq + 2\theta_0)}{x} & -(x + t + \frac{pq + \theta_0}{x}) \end{pmatrix} \\ \mathcal{R}_{IV}(x, t) &= \begin{pmatrix} x + q + t & 1 \\ -2(pq + \theta_0 + \theta_\infty) & -(x + q + t) \end{pmatrix}\end{aligned}\quad (2.4)$$

- For (P_V) :

$$\begin{aligned}\mathcal{D}_V(x, t) &= \begin{pmatrix} \frac{t}{2} + \frac{1}{x} \left(pq + \frac{\theta_0}{2} \right) - \frac{1}{x-1} \left(pq + \frac{\theta_0 + \theta_\infty}{2} \right) & -\frac{pq + \theta_0}{x} + \frac{p + \frac{\theta_0 - \theta_1 + \theta_\infty}{2q}}{x-1} \\ \frac{pq}{x} - \frac{pq^2 + q \frac{\theta_0 + \theta_1 + \theta_\infty}{2}}{x-1} & -\left(\frac{t}{2} + \frac{1}{x} \left(pq + \frac{\theta_0}{2} \right) - \frac{1}{x-1} \left(pq + \frac{\theta_0 + \theta_\infty}{2} \right) \right) \end{pmatrix} \\ \mathcal{R}_V(x, t) &= \begin{pmatrix} \frac{x}{2} - \frac{1}{2t} \left(p(q-1)^2 - \theta_0 + \frac{\theta_0 - \theta_1 + \theta_\infty}{2q} + q \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) & -\frac{1}{t} \left(p(q-1) + \theta_0 - \frac{\theta_0 - \theta_1 + \theta_\infty}{2q} \right) \\ -\frac{q}{t} \left(p(q-1) + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) & -\left(\frac{x}{2} - \frac{1}{2t} \left(p(q-1)^2 - \theta_0 + \frac{\theta_0 - \theta_1 + \theta_\infty}{2q} + q \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) \right) \end{pmatrix}\end{aligned}\quad (2.5)$$

- For (P_{VI}) :

$$\mathcal{D}_{VI}(x, t) = \frac{A_0(t)}{x} + \frac{A_1(t)}{x-1} + \frac{A_t(t)}{x-t}, \quad \mathcal{R}_{VI}(x, t) = -\frac{A_t(t)}{x-t} - \frac{(q-t)(\theta_\infty - 1)}{2t(t-1)}\sigma_3 \quad (2.6)$$

where

$$\begin{aligned}A_0 &= \begin{pmatrix} z_0 + \frac{\theta_0}{2} & -\frac{q}{t} \\ \frac{tz_0(z_0 + \theta_0)}{q} & -(z_0 + \frac{\theta_0}{2}) \end{pmatrix}, \quad A_1 = \begin{pmatrix} z_1 + \frac{\theta_1}{2} & \frac{q-1}{t-1} \\ -\frac{(t-1)z_1(z_1 + \theta_1)}{q-1} & -(z_1 + \frac{\theta_1}{2}) \end{pmatrix} \\ A_t &= \begin{pmatrix} z_t + \frac{\theta_t}{2} & -\frac{q-t}{t(t-1)} \\ \frac{t(t-1)z_t(z_t + \theta_t)}{q-t} & -(z_t + \frac{\theta_t}{2}) \end{pmatrix}, \quad A_\infty = \begin{pmatrix} \frac{\theta_\infty}{2} & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix} = -(A_0 + A_1 + A_t)\end{aligned}$$

Here, $z_0(t)$, $z_1(t)$ and $z_t(t)$ are auxiliary functions of t that can be expressed in terms $q(t)$ and a function $p(t)$ defined by:

$$p = \frac{z_0 + \theta_0}{q} + \frac{z_1 + \theta_1}{q-1} + \frac{z_t + \theta_t}{q-t} \quad (2.7)$$

The explicit expressions of (z_0, z_1, z_t) in terms of (p, q) are given by:

$$\begin{aligned}z_0 &= \frac{1}{\theta_\infty t} \left[q^2(q-1)(q-t)p^2 - pq((\theta_0 + \theta_1 + \theta_t - \theta_\infty)q^2 - ((\theta_0 + \theta_1 - \theta_\infty)t + \theta_0 + \theta_t - \theta_\infty)q + (\theta_0 - \theta_\infty)t) \right. \\ &\quad \left. + \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)^2 q^2 - \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)((\theta_0 + \theta_1 - \theta_t - \theta_\infty)t + \theta_0 - \theta_1 - \theta_\infty + \theta_t)q - t\theta_0\theta_\infty \right] \\ z_1 &= \frac{1}{(t-1)\theta_\infty} \left[-q(q-1)^2(q-t)p^2 + p(q-1)((\theta_0 + \theta_1 + \theta_t - \theta_\infty)^2 q^2 - q((\theta_0 + \theta_1 - \theta_\infty)t + \theta_0 + \theta_t) + \theta_0 t) \right. \\ &\quad \left. - \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)^2 (q-1)^2 + \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)(t(\theta_0 + \theta_1 - \theta_t - \theta_\infty) + 2\theta_\infty - 2\theta_1)(q-1) \right. \\ &\quad \left. - \theta_1\theta_\infty(t-1) \right] \\ z_t &= -z_0 - z_1 - \frac{1}{2}(\theta_0 + \theta_1 + \theta_t + \theta_\infty)\end{aligned}\quad (2.8)$$

In the previous six Lax pair, the monodromy parameters $\theta, \theta_0, \theta_1, \theta_\infty, \theta_t$ are assumed to be generic (i.e. non singular in a sense defined below). In all six cases, the functions $q(t)$ satisfy the corresponding Painlevé equation. The functions $p(t)$ are directly connected to the Hamiltonian formulation of the problems that will be given in the next section. As one can notice, our Lax pairs differ from the original ones given by Jimbo and Miwa. For completeness, we provide the

correspondence in appendix A. The choice of a Lax pair depends on a gauge choice $\Psi(x, t) \rightarrow U(x, t)\Psi(x, t)$. Indeed, for a general gauge transformation $\tilde{\Psi}(x, t) = U(x, t)\Psi(x, t)$ we have that $\tilde{\Psi}(x, t)$ satisfies a Lax pair system $\partial_x \tilde{\Psi} = \tilde{\mathcal{D}}\tilde{\Psi}$, $\partial_t \tilde{\Psi} = \tilde{\mathcal{R}}\tilde{\Psi}$ with:

$$\begin{aligned}\tilde{\mathcal{D}}(x, t) &= U(x, t)\mathcal{D}(x, t)U^{-1}(x, t) + \frac{\partial U}{\partial x}(x, t)U^{-1}(x, t) \\ \tilde{\mathcal{R}}(x, t) &= U(x, t)\mathcal{R}(x, t)U^{-1}(x, t) + \frac{\partial U}{\partial t}(x, t)U^{-1}(x, t)\end{aligned}\quad (2.9)$$

We discuss the possible gauge transformations and their consequences in the next sections:

2.2 Admissible gauge transformations

In general, the Lax pair characterizing the Painlevé equations prescribes the pole singularities at $x \in \{0, 1, t\}$. Therefore, in order to keep this structure, we may only allow gauge transformations in which $U(x, t)$ only depends on t but not on x (i.e. we take $U(t)$) or gauge transformations that depend on x in a trivial way in order not to introduce “fake” singularities. This leads us to introduce the following gauge transformations:

Definition 2.2 (Admissible gauge transformations) *The gauge transformations $\tilde{\Psi}(x, t) = U(x, t)\Psi(x, t)$ where $U(x, t)$ is given by either:*

- $U(x, t) = f(x, t)I_2 = \left(\prod_{i=1}^r (x - a_i(t))^{\nu_i \theta_i} \right) I_2$ where $(a_i(t))_{1 \leq i \leq r}$ are the pole singularities, $(\theta_i)_{1 \leq i \leq r}$ the corresponding monodromy parameters and $(\nu_i)_{1 \leq i \leq r}$ are given real numbers.
- $U(x, t) = U(t)$ is independent of x with $U(t)$ invertible.

are called *admissible*.

The choice of the specific form of $f(x, t) = \prod_{i=1}^r (x - a_i(t))^{\nu_i \theta_i}$ is made so that $x \mapsto U(x, t)$ is invertible except at the existing singularities of the Lax pairs. Note also that it is crucial that the x -dependent gauge transformations are diagonal. Indeed, in the first kind of transformation we get:

$$U(x, t) = \left(\prod_{i=1}^r (x - a_i(t))^{\nu_i \theta_i} \right) I_2 \Rightarrow (\tilde{\mathcal{D}}(x, t), \tilde{\mathcal{R}}(x, t)) = \left(\mathcal{D}(x, t) + \sum_{i=1}^r \frac{\nu_i \theta_i}{x - a_i}, \mathcal{R}(x, t) - \sum_{i=1}^r \frac{\nu_i \theta_i \dot{a}_i}{x - a_i} \right) \quad (2.10)$$

In the second case we get:

$$U(x, t) = U(t) \Rightarrow (\tilde{\mathcal{D}}(x, t), \tilde{\mathcal{R}}(x, t)) = \left(U(t)\mathcal{D}(x, t)U^{-1}(t), U(t)\mathcal{R}(x, t)U^{-1}(t) + \dot{U}(t)U^{-1}(t) \right) \quad (2.11)$$

Note in particular that we only used admissible gauge transformations (in appendix B) to connect the Jimbo-Miwa Lax pairs to ours. As we will see later, **all interesting quantities, including Eynard-Orantin differentials, symplectic invariants, tau-functions, determinantal formulas and spectral curve (up to a trivial symplectic transformation) are invariant under admissible gauge transformations.** For simplicity, we chose the gauge in which all Lax pair are traceless (this fixes the right choice for $f(x, t)$) and in which the matrices $\mathcal{D}(x, t)$ and $\mathcal{R}(x, t)$ have a nice \hbar series expansion (this fixes $U(t)$ up to trivial constant factors). However, we stress again that our main results remain valid in any admissible gauge.

2.3 Introduction of \hbar

We deform the previous Lax pair by introducing a formal parameter \hbar in the Lax system:

Definition 2.3 (Introduction of \hbar) *We define a \hbar -deformation of the Lax pairs by requiring:*

$$\hbar \partial_x \Psi(x, t) = \mathcal{D}(x, t) \Psi(x, t) \text{ and } \hbar \partial_t \Psi(x, t) = \mathcal{R}(x, t) \Psi(x, t) \quad (2.12)$$

In particular the compatibility equation now reads:

$$\hbar \partial_t \mathcal{D}(x, t) - \hbar \partial_x \mathcal{R}(x, t) + [\mathcal{D}(x, t), \mathcal{R}(x, t)] = 0 \quad (2.13)$$

Introducing \hbar in this way may appear arbitrary, but for all six cases we can recover the \hbar -deformation by a proper rescaling. Indeed, if we perform rescaling of the form:

$$(\tilde{t}, \tilde{x}, \tilde{q}, \tilde{p}, \tilde{\theta}_i) = (\hbar^{\delta_t} t, \hbar^{\delta_x} x, \hbar^{\delta_q} q, \hbar^{\delta_p} p, \hbar^{\delta_i} \theta_i) \text{ and } \tilde{\Psi} = \begin{pmatrix} \hbar^{\delta_\Psi} & 0 \\ 0 & \hbar^{-\delta_\Psi} \end{pmatrix} \Psi \quad (2.14)$$

with suitable exponents, then the tilde systems expressed in the tilde variables satisfy the \hbar -deformed versions of the Painlevé systems. Note that as soon as the general form of the rescaling (2.14) is chosen, the choice of exponents giving rise to non-trivial solutions is unique. However, one could imagine other kinds of scalings that may provide interesting regimes as well. In our six cases, the rescalings are given by:

- For Painlevé I, we can obtain the \hbar -deformed version from (2.1) with the change of variables:

$$(\tilde{t}, \tilde{x}, \tilde{q}, \tilde{p}) = \left(\hbar^{\frac{4}{5}} t, \hbar^{\frac{2}{5}} x, \hbar^{\frac{2}{5}} q, \hbar^{\frac{3}{5}} p \right) \text{ and } \tilde{\Psi} = \begin{pmatrix} \hbar^{\frac{1}{10}} & 0 \\ 0 & \hbar^{-\frac{1}{10}} \end{pmatrix} \Psi \quad (2.15)$$

- For Painlevé II, we can obtain the \hbar -deformed version from (2.2) with the change of variables:

$$(\tilde{t}, \tilde{x}, \tilde{q}, \tilde{p}, \tilde{\theta}) = \left(\hbar^{\frac{2}{3}} t, \hbar^{\frac{1}{3}} x, \hbar^{\frac{1}{3}} q, \hbar^{\frac{2}{3}} p, \hbar \theta \right) \text{ and } \tilde{\Psi} = \begin{pmatrix} \hbar^{\frac{1}{6}} & 0 \\ 0 & \hbar^{-\frac{1}{6}} \end{pmatrix} \Psi \quad (2.16)$$

- For Painlevé III, we can obtain the \hbar -deformed version from (2.3) with the change of variables:

$$(\tilde{t}, \tilde{x}, \tilde{q}, \tilde{p}, \tilde{\theta}_0, \tilde{\theta}_\infty) = (\hbar t, x, q, \hbar p, \hbar \theta_0, \hbar \theta_\infty) \text{ and } \tilde{\Psi} = \Psi \quad (2.17)$$

- For Painlevé IV, we can obtain the \hbar -deformed version from (2.4) with the change of variables:

$$(\tilde{t}, \tilde{x}, \tilde{q}, \tilde{p}, \tilde{\theta}_0, \tilde{\theta}_\infty) = \left(\hbar^{\frac{1}{2}} t, \hbar^{\frac{1}{2}} x, \hbar^{\frac{1}{2}} q, \hbar^{\frac{1}{2}} p, \hbar \theta_0, \hbar \theta_\infty \right) \text{ and } \tilde{\Psi} = \begin{pmatrix} \hbar^{\frac{1}{4}} & 0 \\ 0 & \hbar^{-\frac{1}{4}} \end{pmatrix} \Psi \quad (2.18)$$

- For Painlevé V, we can obtain the \hbar -deformed version from (2.5) with the change of variables:

$$(\tilde{t}, \tilde{x}, \tilde{q}, \tilde{p}, \tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_\infty) = (\hbar t, x, q, \hbar p, \hbar \theta_0, \hbar \theta_1, \hbar \theta_\infty) \text{ and } \tilde{\Psi} = \Psi \quad (2.19)$$

- For Painlevé VI, we can obtain the \hbar -deformed version from (2.6) with the change of variables:

$$(\tilde{t}, \tilde{x}, \tilde{q}, \tilde{z}_0, \tilde{z}_1, \tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty) = (t, x, q, \hbar z_0, \hbar z_1, \hbar \theta_0, \hbar \theta_1, \hbar \theta_t, \hbar \theta_\infty) \text{ and } \tilde{\Psi} = \begin{pmatrix} \hbar^{\frac{1}{2}} & 0 \\ 0 & \hbar^{-\frac{1}{2}} \end{pmatrix} \Psi \quad (2.20)$$

In this way, we observe that the introduction of a parameter \hbar is equivalent to consider a suitable scaling limit of the problem in which the monodromy parameters, x and t are sent to 0 or infinity in a certain way. Note also that we recover all standard Lax pairs by taking $\hbar = 1$. Introducing the \hbar parameter modifies the Lax pairs, their compatibility equations and the Painlevé equations. For example the Painlevé 6 Lax pair becomes:

$$\mathcal{D}(x, t) = \frac{A_0(t)}{x} + \frac{A_1(t)}{x-1} + \frac{A_t(t)}{x-t}, \quad \mathcal{R}(x, t) = -\frac{A_t(t)}{x-t} - \frac{(q-t)(\theta_\infty - \hbar)}{2t(t-1)}\sigma_3 \quad (2.21)$$

where

$$\begin{aligned} A_0 &= \begin{pmatrix} z_0 + \frac{\theta_0}{2} & -\frac{q}{t} \\ \frac{tz_0(z_0 + \theta_0)}{q} & -(z_0 + \frac{\theta_0}{2}) \end{pmatrix}, \quad A_1 = \begin{pmatrix} z_1 + \frac{\theta_1}{2} & \frac{q-1}{t-1} \\ -\frac{(t-1)z_1(z_1 + \theta_1)}{q-1} & -(z_1 + \frac{\theta_1}{2}) \end{pmatrix} \\ A_t &= \begin{pmatrix} z_t + \frac{\theta_t}{2} & -\frac{q-t}{t(t-1)} \\ \frac{t(t-1)z_t(z_t + \theta_t)}{q-t} & -(z_t + \frac{\theta_t}{2}) \end{pmatrix}, \quad A_\infty = \begin{pmatrix} \frac{\theta_\infty}{2} & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix} = -(A_0 + A_1 + A_t) \end{aligned}$$

For completeness, we propose in appendix B the derivation of the \hbar -deformed version of the Painlevé equations. We find:

- Painlevé I:

$$\hbar^2 \ddot{q} = 6q^2 + t \quad (2.22)$$

- Painlevé II:

$$\hbar^2 \ddot{q} = 2q^3 + tq + \frac{\hbar}{2} - \theta \quad (2.23)$$

- Painlevé III:

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{q} \dot{q}^2 - \frac{\hbar^2}{t} \dot{q} + \frac{4}{t} (\theta_0 q^2 - \theta_\infty + \hbar) + 4q^3 - \frac{4}{q} \quad (2.24)$$

- Painlevé IV:

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{2q} \dot{q}^2 + 2 \left(3q^3 + 4tq^2 + (t^2 - 2\theta_\infty + \hbar)q - \frac{\theta_0^2}{q} \right) \quad (2.25)$$

- Painlevé V:

$$\begin{aligned} \hbar^2 \ddot{q} &= \left(\frac{1}{2q} + \frac{1}{q-1} \right) (\hbar \dot{q})^2 - \hbar^2 \frac{\dot{q}}{t} + \frac{(q-1)^2}{t^2} \left(\alpha q + \frac{\beta}{q} \right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1} \\ \text{with} \quad \alpha &= \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{8}, \quad \beta = -\frac{(\theta_0 - \theta_1 + \theta_\infty)^2}{8}, \quad \gamma = \theta_0 + \theta_1 - \hbar \text{ and } \delta = -\frac{1}{2} \end{aligned} \quad (2.26)$$

- Painlevé VI:

$$\begin{aligned} \hbar^2 \ddot{q} &= \frac{\hbar^2}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) \dot{q}^2 - \hbar^2 \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) \dot{q} \\ &\quad + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left[\alpha + \beta \frac{t}{q^2} + \gamma \frac{t-1}{(q-1)^2} + \delta \frac{t(t-1)}{(q-t)^2} \right] \end{aligned} \quad (2.27)$$

where the parameters are:

$$\alpha = \frac{1}{2}(\theta_\infty - \hbar)^2, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2} \text{ and } \delta = \frac{\hbar^2 - \theta_t^2}{2}$$

As mentioned earlier we recover the standard Painlevé equations by taking $\hbar = 1$. Introducing the \hbar parameter modifies the admissible gauge transformations in the following sense:

Definition 2.4 (\hbar -deformed admissible gauge transformations) *With the introduction of the parameter \hbar , the admissible gauge transformations $\tilde{\Psi}(x, t) = U(x, t, \hbar)\Psi(x, t)$ are deformed into:*

- $U(x, t, \hbar) = \left(\prod_{i=1}^r (x - a_i(t))^{\frac{\nu_i \theta_i}{\hbar}} \right) I_2$ where $(a_i(t))_{1 \leq i \leq r}$ are the pole singularities, $(\theta_i)_{1 \leq i \leq r}$ the corresponding monodromy parameters and $(\nu_i)_{1 \leq i \leq r}$ are given real numbers independent of x , t and \hbar .
- $U(x, t, \hbar) = U(t, \hbar)$ is independent of x with $U(t, \hbar)$ invertible.

The previous admissible gauge transformations transform the Lax pairs like:

$$U(x, t, \hbar) = \left(\prod_{i=1}^r (x - a_i)^{\frac{\nu_i \theta_i}{\hbar}} \right) I_2 \Rightarrow \left(\tilde{\mathcal{D}}(x, t), \tilde{\mathcal{R}}(x, t) \right) = \left(\mathcal{D}(x, t) + \sum_{i=1}^r \frac{\nu_i \theta_i}{x - a_i}, \mathcal{R}(x, t) - \sum_{i=1}^r \frac{\nu_i \theta_i \dot{a}_i}{x - a_i} \right) \quad (2.28)$$

and

$$U(x, t, \hbar) = U(t, \hbar) \Rightarrow \left(\tilde{\mathcal{D}}(x, t), \tilde{\mathcal{R}}(x, t) \right) = \left(U(t, \hbar) \mathcal{D}(x, t) U^{-1}(t, \hbar), U(t, \hbar) \mathcal{R}(x, t) U^{-1}(t, \hbar) + \hbar \dot{U}(t, \hbar) U^{-1}(t, \hbar) \right) \quad (2.29)$$

Remark 2.5 *One could consider more general gauge transformations of the form $U(x, t, \hbar) = f(x, t, \hbar) I_2$ with any arbitrary function $f(x, t, \hbar)$. In that case, the correlation functions associated to the Lax pair W_n (defined in (6.1) or equivalently in (6.7)) would be invariant under such transformations for $n \geq 2$. However $W_1(x)$ would be changed into $\tilde{W}_1(x) = W_1(x) - \frac{\partial_x f(x, t, \hbar)}{f(x, t, \hbar)}$. This implies that $W_1(x)$ may not have a nice \hbar series expansion and may have poles at zeros of $f(x, t, \hbar)$. On the topological recursion side, the spectral curve (defined in section 5) would be changed by a symplectic transformation of the form $(\tilde{x}, \tilde{Y}) = \left(x, Y - \left(\frac{\partial_x f(x, t, \hbar)}{f(x, t, \hbar)} \right)^{(0)} \right)$. Note that this only makes sense if $\frac{\partial_x f(x, t, \hbar)}{f(x, t, \hbar)}$ admits a series expansion in \hbar . This transformation keeps the Eynard-Orantin differentials $\omega_n^{(g)}$ with $(n, g) \neq (1, 0)$ unchanged (see definition 5.3) as well as the symplectic invariants $F^{(g)}$ for $g \geq 0$. However $\omega_1^{(0)}(x)$ transforms into $\tilde{\omega}_1^{(0)}(z) = \omega_1^{(0)}(z) - \left(\frac{\partial_x f(x, t, \hbar)}{f(x, t, \hbar)} \right)^{(-1)} dx(z)$. Consequently one can see that the correspondence between $\tilde{W}_1(x)$ and $\omega_1^{(0)}(z)$ may only be gauge invariant if $\frac{\partial_x f(x, t, \hbar)}{f(x, t, \hbar)}$ is proportional to $\frac{1}{\hbar}$. This is precisely why we restricted the admissible gauge transformations to this specific form in definition 2.2 (and after rescaling definition 2.4). Note that if we are only interested in the tau-function or in correlation functions W_n with $n \geq 2$ then we may include these general gauge transformations in the definition of admissible gauge transformations.*

2.4 Hamiltonians, tau-function and Okamoto's σ -form of the Painlevé equations

It is well known since the works of Okamoto [34] that the Painlevé equations can be represented as Hamiltonian systems. In this section, we provide the corresponding Hamiltonians associated to our \hbar -deformed version of the Painlevé equations as well as the Jimbo-Miwa τ -functions and Okamoto's σ -functions.

Theorem 2.6 (Hamiltonian formulation) *All six \hbar -deformations of the Painlevé equations can be recovered from an Hamiltonian system $H_J(p, q, \hbar)$ with $1 \leq J \leq 6$ with the \hbar -deformed equations of motion:*

$$\hbar \dot{q} = \frac{\partial H_J}{\partial p}(p, q, \hbar) \text{ and } \hbar \dot{p} = -\frac{\partial H_J}{\partial q}(p, q, \hbar) \quad (2.30)$$

We list here the various Hamiltonians as well as their relations to the tau-function and Okamoto's σ -functions.

- Painlevé I: The Hamiltonian is given by:

$$H_I(p, q, t) = \frac{1}{2}p^2 - 2q^3 - tq \quad (2.31)$$

Moreover we have:

$$\frac{d}{dt} \log \tau_I(t) = H_I(p(t), q(t)) \text{ and } \sigma_I(t) = \frac{d}{dt} \log \tau_I(t) \quad (2.32)$$

Okamoto sigma-function satisfies the following differential equation:

$$\hbar^2 \ddot{\sigma}_I^2 + 4\dot{\sigma}_I^3 + 2t\dot{\sigma}_I - 2\sigma_I = 0 \quad (2.33)$$

- Painlevé II: The Hamiltonian is given by:

$$H_{II}(p, q, t) = \frac{1}{2}p^2 + (q^2 + \frac{t}{2})p + \theta q \quad (2.34)$$

Moreover we have:

$$\frac{d}{dt} \log \tau_{II}(t) = H_{II}(p(t), q(t)) \text{ and } \sigma_{II}(t) = \frac{d}{dt} \log \tau_{II}(t) \quad (2.35)$$

Okamoto sigma-function satisfies the following differential equation:

$$\hbar^2 \ddot{\sigma}_{II}^2 + 4\dot{\sigma}_{II}^3 + 2t\dot{\sigma}_{II}^2 - 2\sigma_{II}\dot{\sigma}_{II} - \frac{\theta^2}{4} = 0 \quad (2.36)$$

- Painlevé III: The Hamiltonian is given by:

$$H_{III}(p, q, t, \hbar) = \frac{1}{t} \left[2q^2p^2 + 2(-tq^2 + \theta_\infty q + t)p - (\theta_0 + \theta_\infty)tq - t^2 - \frac{1}{4}(\theta_0^2 - \theta_\infty^2) - \hbar pq \right] \quad (2.37)$$

Moreover we have:

$$\begin{aligned} \frac{d}{dt} \log \tau_{III}(t) &= H_{III}(p(t), q(t)) + \hbar \frac{pq}{t} \\ &= \frac{1}{t} \left[2q^2p^2 + 2(-tq^2 + \theta_\infty q + t)p - (\theta_0 + \theta_\infty)tq - t^2 - \frac{1}{4}(\theta_0^2 - \theta_\infty^2) \right] \end{aligned} \quad (2.38)$$

Okamoto sigma-function is directly connected to the tau-function by $\sigma_{III}(t) = t \frac{d}{dt} \log \tau_{III}(t)$. It satisfies the following differential equation:

$$\hbar^2 (t\ddot{\sigma}_{III} - \dot{\sigma}_{III})^2 - 4(2\sigma_{III} - t\dot{\sigma}_{III})(\dot{\sigma}_{III}^2 - 4t^2) - 2(\theta_0^2 + \theta_\infty^2)(\dot{\sigma}_{III}^2 + 4t^2) + 16\theta_0\theta_\infty t\dot{\sigma}_{III} = 0 \quad (2.39)$$

- Painlevé IV: The Hamiltonian is given by:

$$H_{IV}(p, q, t) = qp^2 + 2(q^2 + tq + \theta_0)p + 2(\theta_0 + \theta_\infty)q \quad (2.40)$$

Moreover we have:

$$\frac{d}{dt} \log \tau_{IV}(t) = H_{IV}(p(t), q(t)) \text{ and } \sigma_{IV}(t) = \frac{d}{dt} \log \tau_4(t) + 2t \left(\theta_0 + \frac{1}{3} \theta_\infty \right) \quad (2.41)$$

Okamoto sigma-function satisfies the following differential equation:

$$\hbar^2 \ddot{\sigma}_{IV}^2 - 4(t\dot{\sigma}_{IV} - \sigma_{IV})^2 + 4(\dot{\sigma}_{IV} + \alpha)(\dot{\sigma}_{IV} + \beta)(\dot{\sigma}_{IV} + \gamma) = 0 \quad (2.42)$$

with $\alpha = -2\theta_0 - \frac{2}{3}\theta_\infty$, $\beta = 2\theta_0 - \frac{2}{3}\theta_\infty$ and $\gamma = \frac{4}{3}\theta_\infty$.

- Painlevé V: The Hamiltonian is given by:

$$H_V(p, q, t) = \frac{1}{t} \left[q(q-1)^2 p^2 + \left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2} (q-1)^2 + (\theta_0 + \theta_1)q(q-1) - tq \right) p + \frac{1}{2} \theta_0 (\theta_0 + \theta_1 + \theta_\infty) q \right] \quad (2.43)$$

Jimbo-Miwa tau-function is directly connected to the Hamiltonian by:

$$\frac{d}{dt} \log \tau_V = H_V(p(t), q(t)) - \frac{\theta_0 + \theta_\infty}{2} - \frac{1}{4t} (\theta_0 - \theta_1 + \theta_\infty) (\theta_0 + \theta_1 + \theta_\infty) \quad (2.44)$$

Okamoto sigma-function is defined by $\sigma_V(t) = tH_V(p(t), q(t))$ and satisfies:

$$\hbar^2 t^2 \ddot{\sigma}_V^2 - (\sigma_V - t\dot{\sigma}_V + 2\dot{\sigma}_V^2 + (\nu_1 + \nu_2 + \nu_3)\dot{\sigma}_V)^2 + 4\dot{\sigma}_V(\dot{\sigma}_V + \nu_1)(\dot{\sigma}_V + \nu_2)(\dot{\sigma}_V + \nu_3) = 0 \quad (2.45)$$

where $(\nu_1, \nu_2, \nu_3) = \left(-\frac{\theta_0 - \theta_1 + \theta_\infty}{2}, -\theta_0, -\frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right)$.

- Painlevé VI: The Hamiltonian is given by:

$$\begin{aligned} H_{VI}(p, q, t, \hbar) &= \frac{1}{t(t-1)} \left[q(q-1)(q-t)p^2 - p(\theta_0(q-1)(q-t) + \theta_1 q(q-t) + (\theta_t - \hbar)q(q-1)) \right. \\ &\quad \left. + \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)(\theta_0 + \theta_1 + \theta_t + \theta_\infty - \hbar)(q-t) + \frac{1}{2}((t-1)\theta_0 + t\theta_1)(\theta_t - \hbar) \right] \end{aligned} \quad (2.46)$$

Jimbo-Miwa tau-function is defined by:

$$\begin{aligned} \frac{d}{dt} \log \tau_{VI} &= H_{VI}(p(t), q(t), \hbar = 0) \\ &= \frac{1}{t(t-1)} \left[q(q-1)(q-t)p^2 - p(\theta_0(q-1)(q-t) + \theta_1 q(q-t) + \theta_t q(q-1)) \right. \\ &\quad \left. + \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)(\theta_0 + \theta_1 + \theta_t + \theta_\infty)(q-t) + \frac{1}{2}((t-1)\theta_0 + t\theta_1)\theta_t \right] \end{aligned} \quad (2.47)$$

We can then define Okamoto sigma-function with:

$$\sigma_{VI}(t) = t(t-1) \frac{d}{dt} \log \tau_{VI}(t) + \frac{1}{4}(\theta_t^2 - \theta_\infty^2)t - \frac{1}{8}(\theta_0^2 + \theta_t^2 - \theta_1^2 - \theta_\infty^2) \quad (2.48)$$

It satisfies the following differential equation:

$$0 = \hbar^2 \dot{\sigma}_{VI} t^2 (t-1)^2 \ddot{\sigma}_{VI}^2 + \left(2\dot{\sigma}_{VI}(t\dot{\sigma}_{VI} - \sigma_{VI}) - \dot{\sigma}_{VI}^2 + \frac{1}{16}(\theta_t^2 - \theta_\infty^2)(\theta_1^2 - \theta_0^2) \right)^2$$

$$(2.49) \quad - \left(\dot{\sigma}_{VI} + \frac{(\theta_t + \theta_\infty)^2}{4} \right) \left(\dot{\sigma}_{VI} + \frac{(\theta_t - \theta_\infty)^2}{4} \right) \left(\dot{\sigma}_{VI} + \frac{(\theta_0 + \theta_1)^2}{4} \right) \left(\dot{\sigma}_{VI} + \frac{(\theta_0 - \theta_1)^2}{4} \right)$$

Note that we can also define $y(t) = t(t-1)H_{VI}(p(t), q(t), \hbar) + \frac{1}{4}((\theta_t - \hbar)^2 - (\theta_\infty - \hbar)^2)t - \frac{1}{8}(\theta_0^2 - \theta_1^2 + (\theta_t - \hbar)^2 - (\theta_\infty - \hbar)^2)$ and observe that it satisfies:

$$(2.50) \quad \begin{aligned} 0 = & \hbar^2 t^2 (t-1)^2 \dot{y} \ddot{y}^2 + \left(2\dot{y}(t\dot{y} - y) - \dot{y}^2 + \frac{1}{16}(\theta_t - \theta_\infty)(\theta_t + \theta_\infty - 2\hbar)(\theta_1^2 - \theta_0^2) \right)^2 \\ & - \left(\dot{y} + \frac{(\theta_t + \theta_\infty - 2\hbar)^2}{4} \right) \left(\dot{y} + \frac{(\theta_t - \theta_\infty)^2}{4} \right) \left(\dot{y} + \frac{(\theta_0 + \theta_1)^2}{4} \right) \left(\dot{y} + \frac{(\theta_0 - \theta_1)^2}{4} \right) \end{aligned}$$

One can verify that the equations of motion for these Hamiltonians respectively recover (B.1), (B.2), (B.3), (B.4), (B.5), (B.6). Note that the tau functions $\ln \tau_J$ are defined up to constants (in the sense independent of t) since they are defined through their derivatives. We also remark:

1. Only Painlevé III and Painlevé VI Hamiltonians explicitly depend on \hbar .
2. In all six cases the tau-function is recovered from the Hamiltonian by taking:

$$\frac{d}{dt} \log \tau_J(t) = H_J(p(t), q(t), t, \hbar = 0) \quad (2.51)$$

Consequently all tau-functions can be seen as functions of (q, p) without any explicit dependence on \hbar .

3. The σ -functions and the tau-functions always satisfy differential equations that only involve \hbar^2 but not directly \hbar .

3 Formal series expansion in \hbar

3.1 General assumption

In this paper we are interested in the computation of the formal series expansion in \hbar . This is equivalent to the WKB expansion of the matrix $\Psi(x, t)$. It is well known that these series expansions may not be convergent and that convergent solutions may require additional correction terms. In this paper we only deal with the combinatorial formal series expansion in \hbar and postpone the convergence issues to future works. Consequently we assume that the solutions of the Painlevé equations admit a series expansion in \hbar :

Assumption 3.1 (Existence of a formal expansion) *We assume that the solutions $(q_J(t))_{1 \leq J \leq 6}$ of the Painlevé equations admit (possibly formal) series expansions in \hbar of the form:*

$$q_J(t) = \sum_{k=0}^{\infty} q_J^{(k)}(t) \hbar^k \quad (3.1)$$

Moreover to have more compact notation we will denote $q^{(0)}(t)$ by $q_0(t)$ in the rest of the paper.

Under the previous assumption, it is very easy to see that the other quantities in the Lax pairs (2.1)~(2.6) admit a series expansion of the same form. Thus we get:

$$\mathcal{D}_J(x, t) = \sum_{k=0}^{\infty} \mathcal{D}_J^{(k)}(x, t) \hbar^k \quad \text{and} \quad \mathcal{R}_J(x, t) = \sum_{k=0}^{\infty} \mathcal{R}_J^{(k)}(x, t) \hbar^k \quad (3.2)$$

We stress here **that the existence of a series expansion of the form (3.2) depends on the very specific selection of the gauge $U(t, \hbar)$.** However the existence of such series expansions is invariant under gauge transformations defined in (2.28).

Remark 3.2 *We note that the gauge choice in [23] is not exactly of the right form to obtain (3.2). Indeed, in Jimbo-Miwa, most of the Lax pairs involved off-diagonal factors $u(t), v(t)$ or $k(t)$ satisfying differential equations of the form $\hbar \frac{d \log u}{dt} = \dots$ where \dots indicates some explicit functions of $(q(t), p(t))$ having a series expansion starting at \hbar^0 . Thus $u(t)$ (resp. $v(t), k(t)$) would have a term in $e^{-\frac{u^{(-1)}(t)}{\hbar}}$. Consequently in Jimbo-Miwa gauge we would get series expansions of the form:*

$$\begin{aligned} \mathcal{D}_J(x, t) &= \begin{pmatrix} \sum_{k=0}^{\infty} d_{1,1}(x, t) & e^{-\frac{u^{(-1)}(t)}{\hbar}} \left(\sum_{k=0}^{\infty} d_{1,2}(x, t) \right) \\ e^{\frac{u^{(-1)}(t)}{\hbar}} \left(\sum_{k=0}^{\infty} d_{2,1}(x, t) \right) & \sum_{k=0}^{\infty} d_{2,2}(x, t) \end{pmatrix} \\ \mathcal{R}_J(x, t) &= \begin{pmatrix} \sum_{k=0}^{\infty} r_{1,1}(x, t) & e^{-\frac{u^{(-1)}(t)}{\hbar}} \left(\sum_{k=0}^{\infty} r_{1,2}(x, t) \right) \\ e^{\frac{u^{(-1)}(t)}{\hbar}} \left(\sum_{k=0}^{\infty} r_{2,1}(x, t) \right) & \sum_{k=0}^{\infty} r_{2,2}(x, t) \end{pmatrix} \end{aligned} \quad (3.3)$$

In all six cases, we selected the gauge by an adequate transformation $\Psi_J(x, t) \rightarrow U_J(t, \hbar) \Psi_J(x, t)$ with $U_J(t, \hbar) = \begin{pmatrix} u^{(-1)}(t)^{\frac{1}{2}} & 0 \\ 0 & u^{(-1)}(t)^{-\frac{1}{2}} \end{pmatrix}$ such that the exponential factors vanish. Details can be found in appendix A for all six cases. As mentioned earlier, the gauge choice $U(t, \hbar)$ is irrelevant since all interesting quantities that we are about to define will be invariant under admissible gauge transformations.

3.2 Generic monodromy parameters and singular times

In our paper, we want to study generic cases of the Painlevé equations. This implies that some values of the monodromy parameters should be avoided because they correspond to singular cases where the Painlevé equations are degenerate. We assume that:

Assumption 3.3 (Non-singular monodromy parameters) *The monodromy parameters of the Painlevé equations are assumed to be generic in the following sense:*

- For Painlevé II: $\theta \neq 0$
- For Painlevé III: $\theta_{\infty} \neq 0$, $\theta_0 \neq 0$ and $\theta_{\infty}^2 \neq \theta_0^2$
- For Painlevé IV: $\theta_{\infty} \neq 0$, $\theta_0 \neq 0$ and $\theta_{\infty}^2 \neq \theta_0^2$
- For Painlevé V: $\theta_0, \theta_1, \theta_{\infty} \neq 0$ and $\theta_{\infty} + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 \neq 0$ for all possible choice of (ϵ_0, ϵ_1) in $\{-1, 1\}^2$.

- For Painlevé VI: $\theta_0, \theta_1, \theta_t, \theta_\infty \neq 0$ and $\theta_0^2 \neq \theta_1^2$ and $\theta_\infty + \epsilon_0\theta_0 + \epsilon_1\theta_1 \neq 0$ for all possible choice of (ϵ_0, ϵ_1) in $\{-1, 1\}^2$ and $\theta_\infty + \epsilon_0\theta_0 + \epsilon_1\theta_1 + \epsilon_t\theta_t \neq 0$ for all possible choice of $(\epsilon_0, \epsilon_1, \epsilon_t)$ in $\{-1, 1\}^3$.

Moreover, we also need to exclude some specific times t_0 . Let us denote generically:

$$\hbar^2 \ddot{q}(t) = \hbar^2 B_J(q, \dot{q}, t) + A_J(q, t, \hbar) \quad (3.4)$$

the J^{th} Painlevé equations where A_J and B_J are polynomial functions. Note that A_J is always at most linear in \hbar . Then we define singular times in the following sense:

Definition 3.4 (Singular times) *We call a time t singular for the Painlevé equation \mathcal{P}_J if it belongs to the set*

$$\Delta_J = \left\{ t \in \mathbb{C} \text{ such that there exists } q \text{ satisfying } A_J(q, t, \hbar)|_{\hbar=0} = 0 \text{ and } \frac{\partial A_J}{\partial q}(q, t, \hbar)|_{\hbar=0} = 0 \right\}. \quad (3.5)$$

Singular times correspond to specific algebraic relations satisfied by q_0, t and the monodromy parameters. We list here the corresponding conditions on the monodromy parameters to avoid singular cases:

- For Painlevé I: $6q_0^2 + t = 0$ is always satisfied by $q_0(t)$. The condition of singular times adds the relation $12q_0 = 0$ thus giving that only $t = 0$ is a singular time corresponding to $q_0 = 0$.
- For Painlevé II: $2q_0^3 + tq_0 - \theta = 0$ is always satisfied by $q_0(t)$. The condition of singular times adds the relation $6q_0^2 + t = 0$ thus giving that singular times correspond to times for which we have the algebraic relation:

$$4q_0^3 + \theta = 0 \quad (3.6)$$

- For Painlevé III: $tq_0^4 + \theta_0q_0^3 - \theta_\infty q_0 - t = 0$ is always satisfied by $q_0(t)$. The condition of singular times adds the relation $4tq_0^3 + 3\theta_0q_0^2 - \theta_\infty = 0$ thus giving that singular times correspond to times for which we have the algebraic relation:

$$\theta_0q_0^6 - 3\theta_\infty q_0^4 + 3\theta_0q_0^2 - \theta_\infty = 0 \quad (3.7)$$

- For Painlevé IV: $3q_0^4 + 4tq_0^3 + (t^2 - 2\theta_\infty)q_0^2 - \theta_0^2 = 0$ is always satisfied by $q_0(t)$. The condition of singular times adds the relation $6q_0^2 + 6tq_0 + t^2 - 2\theta_\infty = 0$ thus giving that singular times correspond to times for which we have the algebraic relation:

$$3q_0^8 + 8\theta_\infty q_0^6 + 6\theta_0^2 q_0^4 + \theta_0^4 = 0 \quad (3.8)$$

- For Painlevé V: $q_0(t)$ satisfies the equation:

$$\begin{aligned} 0 = & (\theta_0 - \theta_1 - \theta_\infty)^2 q_0^5 - 3(\theta_0 - \theta_1 - \theta_\infty)^2 q_0^4 \\ & - 2(2t^2 - 4(\theta_0 + \theta_1)t - \theta_0^2 - \theta_1^2 - \theta_\infty^2 + 4\theta_\infty(\theta_0 - \theta_1) + 2\theta_0\theta_1)q_0^3 \\ & - 2(2t^2 + 4(\theta_0 + \theta_1)t - \theta_0^2 - \theta_1^2 - \theta_\infty^2 - 4\theta_\infty(\theta_0 - \theta_1) + 2\theta_0\theta_1)q_0^2 \\ & - 3(\theta_0 - \theta_1 + \theta_\infty)^2 q_0 + (\theta_0 - \theta_1 + \theta_\infty)^2 \end{aligned} \quad (3.9)$$

The condition of singular times adds the relation:

$$t = -\frac{(q_0 - 1)^2 ((\theta_0 - \theta_1 - \theta_\infty)^2 q_0^4 + 2(\theta_0 - \theta_1 - \theta_\infty)^2 q_0^3 - 2(\theta_0 - \theta_1 + \theta_\infty)^2 q_0 - (\theta_0 - \theta_1 + \theta_\infty)^2)}{8q_0^3(\theta_0 + \theta_1)}$$

thus giving that singular times correspond to times for which we have the algebraic relation:

$$\begin{aligned} 0 = & (\theta_0 - \theta_1 - \theta_\infty)^4 q_0^9 + 3(\theta_0 - \theta_1 - \theta_\infty)^4 q_0^8 + 8(\theta_0 - \theta_1 - \theta_\infty)^2 (\theta_0^2 + 6\theta_0\theta_1 + \theta_1^2 - \theta_\infty^2) q_0^6 \\ & - 6(\theta_0 - \theta_1 - \theta_\infty)^2 (\theta_0 - \theta_1 + \theta_\infty)^2 q_0^5 + 6(\theta_0 - \theta_1 - \theta_\infty)^2 (\theta_0 - \theta_1 + \theta_\infty)^2 q_0^4 \\ & - 8(\theta_0 - \theta_1 + \theta_\infty)^2 (\theta_0^2 + 6\theta_0\theta_1 + \theta_1^2 - \theta_\infty^2) q_0^3 - 3(\theta_0 - \theta_1 + \theta_\infty)^4 q_0 - (\theta_0 - \theta_1 + \theta_\infty)^4 \end{aligned} \quad (3.10)$$

- For Painlevé VI: $q_0(t)$ satisfies the equation:

$$\theta_\infty^2 - \frac{\theta_0^2 t}{q_0^2} + \frac{(t-1)\theta_1^2}{(q_0-1)^2} - \frac{\theta_t^2 t(t-1)}{(q_0-t)^2} = 0 \quad (3.11)$$

This is equivalent to a polynomial equation of degree 6:

$$\begin{aligned} 0 = & \theta_\infty^2 q_0^6 - 2(t+1)\theta_\infty^2 q_0^5 + (-t\theta_0^2 + (t-1)\theta_1^2 + (t^2 + 4t + 1)\theta_\infty^2 - t(t-1)\theta_t^2) q_0^4 \\ & + 2t((t+1)\theta_0^2 - (t-1)\theta_1^2 - (t+1)\theta_\infty^2 + (t-1)\theta_t^2) q_0^3 \\ & - t((t^2 + 4t + 1)\theta_0^2 - t(t-1)\theta_1^2 - t\theta_\infty^2 + (t-1)\theta_t^2) q_0^2 \\ & + 2t^2(t+1)\theta_0^2 q_0 - t^3\theta_0^2 \end{aligned} \quad (3.12)$$

The condition of singular times adds the relation:

$$\frac{\theta_0^2 t}{q_0^3} - \frac{(t-1)\theta_1^2}{(q_0-1)^3} + \frac{t(t-1)\theta_t^2}{(q_0-t)^3} = 0 \quad (3.13)$$

4 Symmetry $\hbar \leftrightarrow -\hbar$

In this section we are interested into changing the formal parameter \hbar into $-\hbar$ and discuss about the consequences on the various functions $p(t), q(t), \log \tau(t), \sigma(t)$. We denote \dagger the involution operator that changes \hbar into $-\hbar$ and we would like to connect (p^\dagger, q^\dagger) to (p, q) . For example in Painlevé 2, $q^\dagger(t)$ corresponds to the solution of the differential equation $\hbar^2 \ddot{y} = 2y^3 + ty(-\frac{\hbar}{2} - \theta)$ while $q(t)$ corresponds to the solution of the differential equation $\hbar^2 \ddot{y} = 2y^3 + ty(\frac{\hbar}{2} - \theta)$. There are many ways to compute the connection: one can use the fact that $\sigma(t)$ satisfies a differential equation only involving \hbar^2 but not directly \hbar and show recursively that odd coefficients of the series are vanishing. We choose here a simpler way starting directly from the Hamiltonian formalism. More specifically we use the fact that by definition of an Hamiltonian system:

$$H_J(p^\dagger, q^\dagger, t, -\hbar) = H_J(p, q, t, \hbar) \text{ and } \frac{\partial H_J}{\partial t}(p^\dagger, q^\dagger, t, -\hbar) = \frac{\partial H_J}{\partial t}(p, q, t, \hbar) \quad (4.1)$$

The last conditions are sufficient to determine p^\dagger and q^\dagger . Then all other dag quantities $((\log \tau)^\dagger(t)$ or $\sigma^\dagger(t)$ for example) can easily be obtained. We find the following results:

- Painlevé I:

$$q^\dagger = q, p^\dagger = -p, \sigma_1^\dagger = \sigma_1, \left(\frac{d}{dt} \log \tau_1\right)^\dagger = \frac{d}{dt} \log \tau_1 \quad (4.2)$$

- Painlevé II:

$$q^\dagger = -q - \frac{\theta}{p}, p^\dagger = p, \sigma_2^\dagger = \sigma_2, \left(\frac{d}{dt} \log \tau_2\right)^\dagger = \frac{d}{dt} \log \tau_2 \quad (4.3)$$

- Painlevé III:

$$q^\dagger = \frac{-2qp^2 + 2(tq - \theta_\infty)p + t(\theta_0 + \theta_\infty)}{2(p-t)p}, p^\dagger = p, \sigma_3^\dagger = \sigma_3, \left(\frac{d}{dt} \log \tau_3\right)^\dagger = \frac{d}{dt} \log \tau_3 \quad (4.4)$$

- Painlevé IV:

$$q^\dagger = \frac{p(pq + 2\theta_0)}{2(pq + \theta_0 + \theta_\infty)}, p^\dagger = \frac{2q(pq + \theta_0 + \theta_\infty)}{pq + 2\theta_0}, \sigma_4^\dagger = \sigma_4, \left(\frac{d}{dt} \log \tau_4\right)^\dagger = \frac{d}{dt} \log \tau_4 \quad (4.5)$$

- Painlevé V:

$$\begin{aligned} q^\dagger &= \frac{p(2pq + \theta_0 - \theta_1 + \theta_\infty)}{(pq + \theta_0)(2pq + \theta_0 + \theta_1 + \theta_\infty)}, p^\dagger = \frac{q(pq + \theta_0)(2pq + \theta_0 + \theta_1 + \theta_\infty)}{2pq + \theta_0 - \theta_1 + \theta_\infty} \\ \sigma_5^\dagger &= \sigma_5, \left(\frac{d}{dt} \log \tau_5\right)^\dagger = \frac{d}{dt} \log \tau_5 \end{aligned} \quad (4.6)$$

- Painlevé VI:

$$\begin{aligned} q^\dagger &= \frac{t^2 z_0(z_0 + \theta_0)(q-1)}{t^2 z_0(z_0 + \theta_0)(q-1) - (t-1)^2 z_1(z_1 + \theta_1)q}, p^\dagger = \frac{z_0 + \theta_0}{q^\dagger} + \frac{z_1 + \theta_1}{q^\dagger - 1} + \frac{z_t + \theta_t}{q^\dagger - t} \\ z_0^\dagger &= z_0, z_1^\dagger = z_1, z_t^\dagger = z_t, \sigma_6^\dagger = \sigma_6, \left(\frac{d}{dt} \log \tau_6\right)^\dagger = \frac{d}{dt} \log \tau_6 \end{aligned} \quad (4.7)$$

where z_0, z_1 and z_t are given in (2.8)

We then observe in all six cases that the τ -functions and the σ -functions are even function of \hbar . Consequently their series expansions may only involve even powers of \hbar and we may write:

$$\frac{d}{dt} \log \tau_J(t) = \sum_{k=0}^{\infty} \frac{d}{dt} \tau_J^{(2k)} \hbar^{2k} \text{ and } \sigma_J(t) = \sum_{k=0}^{\infty} \sigma_J^{(2k)} \hbar^{2k} \quad (4.8)$$

This is of course consistent with the fact that the differential equations for $\sigma_J(t)$ (equations (2.33)-(2.49)) only involve \hbar^2 but not directly \hbar . This is also coherent with the fact that we want to match the tau-function with the symplectic invariants (that only involve even powers of \hbar) arising from the topological recursion.

5 Spectral curves and topological recursion

5.1 Computation of the spectral curves

Following the theory developed by Bergère and Eynard in [3], the spectral curve associated to a 2×2 Lax pair is defined by:

$$(\det(YI_2 - \mathcal{D}(x, t)))^{(0)} = 0 \quad (5.1)$$

In other words, it is given by the leading order in \hbar of the characteristic polynomial of $\mathcal{D}(x, t)$. In our case, since the Lax pairs are traceless, it is equivalent to an hyper-elliptic equation:

$$Y^2 = -(\det \mathcal{D}_J)^{(0)}(x, t) = E_J(x, t) \text{ with } E_J(x, t) \text{ rational function of } x \quad (5.2)$$

Precisely speaking, the curve (5.1) is a family of algebraic curves parametrized by the time t and the monodromy parameters θ_* , but we will omit the dependence of parameters for simplicity. As presented in [3] and in [16], the spectral curve is the key element to implement the topological recursion. It is also connected with the WKB expansion of the matrix $\Psi_J(x, t)$ since the phase function $s_J(x, t)$ satisfies $s_J(x, t)^2 = E_J(x, t)$. and thus corresponds to the semi-classical limit in the WKB analysis. Computing the spectral curves is relatively straightforward. One must project the compatibility equations at order \hbar^0 . This implies that $q_0(t)$ satisfies a polynomial equation that we provide in appendix C. Then one simply needs to compute the determinant and factorize it using the various identities obtained at order \hbar^0 . We provide here the list of spectral curves for our Painlevé Lax pairs.

$$\begin{aligned}
(P_I) & : Y^2 = 4(x + 2q_0)(x - q_0)^2 \\
(P_{II}) & : Y^2 = (x - q_0)^2 \left(x^2 + 2q_0x + q_0^2 + \frac{\theta}{q_0} \right) \\
(P_{III}) & : Y^2 = \frac{(\theta_\infty - \theta_0 q_0^2)^2 (q_0 x + 1)^2 (x^2 + \frac{2q_0(\theta_\infty q_0^2 - \theta_0)}{\theta_0 q_0^2 - \theta_\infty} x + q_0^2)}{4x^4 (q_0^4 - 1)^2} \text{ with } t = \frac{q_0(\theta_\infty - \theta_0 q_0^2)}{q_0^4 - 1} \\
(P_{IV}) & : Y^2 = \frac{(x - q_0)^2 \left(x^2 + 2(q_0 + t)x + \frac{\theta_0^2}{q_0^2} \right)}{x^2} \text{ where } t = -2q_0 + \sqrt{q_0^2 + 2\theta_\infty + \frac{\theta_0^2}{q_0}} \\
(P_V) & : Y^2 = \frac{t^2(x - Q_0)^2(x - Q_1)(x - Q_2)}{4x^2(x - 1)^2} \\
& \quad \text{(See appendix D for formulas connecting } (Q_0, Q_1, Q_2) \text{ with } q_0, p_0 \text{ and } t) \\
(P_{VI}) & : Y^2 = \frac{\theta_\infty^2(x - q_0)^2 P_2(x)}{4x^2(x - 1)^2(x - t)^2} \text{ with } P_2(x) = x^2 + \left(-1 - \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2} + \frac{\theta_0^2(t - 1)^2}{\theta_\infty^2(q_0 - 1)^2} \right) x + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}
\end{aligned}$$

Table 1: List of all spectral curves

Remark 5.1 As one can see, the spectral curve is only defined through the matrix $\mathcal{D}(x, t)$ but does not involve the second matrix $\mathcal{R}(x, t)$. Consequently, the notion of spectral curve is well-defined not only for Lax pairs but simply for any linear differential system $\partial_x \Psi(x) = L(x) \Psi(x)$. Moreover, an important point is that **the spectral curve is invariant under admissible gauge transformations of the form** $\tilde{\Psi}(x, t) = U(t, \hbar) \Psi(x, t)$ even if $\mathcal{D}^{(0)}(x, t)$ cannot be defined (but $(\det \mathcal{D})^{(0)}$ always will be). Note however that the spectral curve is not invariant under admissible gauge transformations (2.28). Nevertheless, the two spectral curves are connected via a symplectic transformation $(\tilde{x} = F(x, Y), \tilde{Y} = G(x, Y))$ with $d\tilde{x} \wedge d\tilde{Y} = dx \wedge dY$ of the form $\tilde{x} = x$ and $\tilde{Y} = Y + g(x)$ (with $g(x) = -\frac{1}{2} \sum_{i=1}^r \frac{\nu_i \theta_i}{x - a_i}$) under which Eynard-Orantin differentials $\omega_n^{(g)}$ with $(n, g) \neq (1, 0)$ and symplectic invariants (defined in section 5.3) are known to be invariant. Note that the result still holds for the symplectic invariants even if as shown in [17] the $F^{(2g)}$'s presented here are not invariant under the exchange $x \leftrightarrow y$.

5.2 General features of the spectral curves

In the topological recursion, the number and type of branchpoints of the spectral curve are crucial and thus we need to exclude all possible non-generic cases that may arise in the previous list of spectral curves. In this section, we detail why we can exclude all these cases for our spectral curves.

- For PI, the spectral curve presents a double zero at $x = q_0$ and a simple zero at $x = -2q_0$. Consequently as soon as $q_0 \neq 0$, i.e. $t \neq 0$ (since $6q_0^2 + t = 0$) these zeros are distinct. This exceptional situation precisely corresponds to the singular time defined in 3.4.

- For PII, the spectral curve presents a double zero at $x = q_0$ and generically two simple zeros at $x = -q_0 \pm \sqrt{\frac{\theta}{q_0}}$. Thus we observe that these simple zeros are always distinct as soon as the monodromy parameter θ is not vanishing. Moreover, these simple zeros can never equal q_0 as soon as $t \notin \Delta_{II}$. Indeed, saying that one of the simple zero equals the double zero q_0 at time t is equivalent to say that $4q_0^3 + \theta = 0$ which precisely corresponds to the singular times as defined in 3.4.
- For PIII, The spectral curve has a double zero at $x = -\frac{1}{q_0}$ and generically two simple zeros solutions of $(\theta_\infty - \theta_0 q_0^2)X^2 - 2q_0(\theta_\infty q_0^2 - \theta_0)X + q_0^2(\theta_\infty - \theta_0 q_0^2) = 0$. The spectral curve is also singular at $x = 0$. Requiring that we have at least a triple zero is equivalent to having a singular time (3.7). Moreover, requiring that the two simple poles coincide is equivalent to $q_0^2(q_0^4 - 1)(\theta_\infty^2 - \theta_0^2) = 0$. This precisely corresponds to singular monodromy parameters and therefore can be discarded.
- For PIV, the spectral curve presents a double zero at $x = q_0$, a double pole at $x = 0$ and generically two simple zeros satisfying $X^2 + 2(q_0 + t)X + \frac{\theta_0^2}{q_0^2}$. Requiring that a zero occurs at $x = 0$ is equivalent to having the singular monodromy $\theta_0 = 0$. It may happen that for some time t , the two simple zeros coincide. This is equivalent to requiring that $q_0^2(q_0 + t)^2 = \theta_0^2$. However, q_0 satisfies (C.4) at all time which is equivalent to $t^2 q_0^2 + 4t q_0^3 + 3q_0^4 - 2\theta_\infty q_0^2 - \theta_0 = 0$. Combining both equations leads to:

$$(\theta_0 + \theta_\infty)q_0^2 = 0 \text{ or } (\theta_0 - \theta_\infty)q_0^2 = 0$$

Since q_0 can never vanish, it is equivalent to $\theta_0^2 \neq \theta_\infty^2$ i.e. that we have singular monodromy parameters. Eventually requiring that for some time t , one of the simple zero coincide with the double zero q_0 is equivalent to require that $3q_0^3 + 2tq_0 + \frac{\theta_0^2}{q_0} = 0$ which is precisely equivalent to (3.8), i.e. that we have a singular time.

- For PV, the generic case corresponds to a spectral curve with two double poles at $x \in \{0, 1\}$ with a double zero at $x = Q_0$ and two simple zeros at Q_1 and Q_2 . In appendix D, the discussion leads to the fact that Q_0, Q_1 and Q_2 cannot be equal to 0 or 1 as soon as the monodromies θ_0, θ_1 and θ_∞ are non-vanishing. Moreover we also prove that the simple zeros can never coincide as soon as the monodromy parameters are non-singular. Eventually, requiring that we have a triple zero is proved to be equivalent to having a singular time.
- For PVI, the generic case corresponds to a spectral with a double zero at $x = q_0$ and two simple zeros solutions of $P_2(X) = X^2 + \left(-1 - \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2} + \frac{\theta_1^2 (t-1)^2}{\theta_\infty^2 (q_0-1)^2}\right)X + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}$. Note that the polynomial $P_2(x)$ is equivalently defined through the following conditions:

$$P_2(0) = \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}, \quad P_2(1) = \frac{(t-1)^2 \theta_1^2}{\theta_\infty^2 (q_0-1)^2} \text{ and } P_2(t) = \frac{t^2 (t-1)^2 \theta_t^2}{\theta_\infty^2 (q_0-t)^2}. \quad (5.3)$$

So that we have:

$$P_2(x) = \frac{\theta_0^2 t}{\theta_\infty^2 q_0^2} (x-1)(x-t) - \frac{(t-1)\theta_1^2}{\theta_\infty^2 (q_0-1)^2} x(x-t) + \frac{t(t-1)\theta_t^2}{\theta_\infty^2 (q_0-t)^2} x(x-1) \quad (5.4)$$

The spectral curve also has double poles at $x \in \{0, 1, t\}$. Discussion in appendix D leads to the fact that as soon as the monodromy parameters are non-singular then the zeros can never equal the poles $\{0, 1, t\}$ and that the simple zeros may never coincide. Eventually, requiring a triple zero is equivalent to having a singular time (3.13).

In the end, we obtain the following statement:

Proposition 5.2 (Non degeneracy of the spectral curve) *For any $J = \text{I}, \dots, \text{VI}$, the spectral curves generically have a double zero and two simple zeros. Moreover:*

- *As soon as the monodromy parameters are non-singular then the zeros of the spectral curve are different from its poles and the simple zeros are distinct.*
- *Requiring that the spectral curve has a triple zero is equivalent to having a singular time.*

In conclusion as soon as we take non-singular monodromies and avoid singular times then the spectral curves are generic.

5.3 Topological recursion

In the previous list of spectral curves, we observe that the r.h.s. always have at most two simple zeros. This is equivalent to say that the Riemann surfaces \mathcal{S} defined by the spectral curves are of genus 0 and therefore that the functions (x, Y) can be parametrized with rational functions $x(z)$, $y(z)$ of z with $z \in \mathbb{C} \cup \{\infty\}$. The standard approach is to take the Zhukovski parametrization whose expression is given by:

- For (P_I) (one branchpoint case) we can use the parametrization:

$$\begin{cases} x(z) = z^2 - 2q_0 \\ Y(z) = 2z(z^2 - 3q_0). \end{cases} \quad (5.5)$$

- For (P_{II}) , (P_{III}) , (P_{IV}) , (P_V) and (P_{VI}) (two branchpoints cases), let us denote generically $Y^2(x) = (x-a)(x-b)C^2(x)$ with $C(x)$ a rational function of x . Then we can use the general parametrization:

$$\begin{cases} x(z) = \frac{a+b}{2} + \frac{b-a}{4}\left(z + \frac{1}{z}\right) \\ Y(z) = \frac{b-a}{4}\left(z - \frac{1}{z}\right)C(x(z)). \end{cases} \quad (5.6)$$

In the case of the Painlevé equations, the functions $C(x(z))$ are given by:

$$\begin{aligned} C_{II}(x) &= (x - q_0), \quad C_{III}(x) = \frac{(\theta_\infty - \theta_0 q_0^2)(q_0 x + 1)}{2x^2(q_0^4 - 1)}, \quad C_{IV}(x) = \frac{x - q_0}{x} \\ C_V(x) &= \frac{t(x - q_0)}{2x(x - 1)}, \quad C_{VI}(x) = \frac{\theta_\infty(x - q_0)}{2x(x - 1)(x - t)} \end{aligned} \quad (5.7)$$

In both cases, points where the one-form $dx(z)$ is vanishing are called branchpoints. With the parametrizations presented above they are located at $z = 0$ and $z = \pm 1$. In both cases, there exists a global involution $z \mapsto \bar{z}$ satisfying:

$$x(\bar{z}) = x(z) \text{ and } Y(\bar{z}) = Y(z)$$

In the parametrizations presented above, the involution is respectively $\bar{z} = -z$ for Painlevé 1 and $\bar{z} = \frac{1}{z}$ for the other cases. Note that in the case of a genus 0 curve, the involution is not only local (i.e. valid around branchpoints) but global (i.e. valid on the whole Riemann surface). We now recall the definition of correlation functions and symplectic invariants as introduced by Eynard and Orantin in [16].

Definition 5.3 (Definition 4.2 of [16]) For $g \geq 0$ and $n \geq 1$, Eynard-Orantin differentials (known also as correlation functions) $\omega_n^{(g)}(z_1, \dots, z_n)$ of type (g, n) associated to the spectral curve $(x(z), Y(z))$ are defined by the following recursive relations:

$$\omega_1^{(0)}(z_1) = (Y(z_1) - Y(\bar{z}_1))dx(z_1) = 2Y(z_1)dx(z_1), \quad (5.8)$$

$$\omega_2^{(0)}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad (5.9)$$

$$\begin{aligned} \omega_{n+1}^{(g)}(z_0, z_1, \dots, z_n) &= \sum_{r \text{ branchpoints}} \operatorname{Res}_{z \rightarrow r} K(z_0, z) \left[\omega_{n+1}^{(g-1)}(z, \bar{z}, z_1, \dots, z_n) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I \cup J = \{1, \dots, n\}}} \omega_{1+|I|}^{(g_1)}(z, z_I) \omega_{1+|J|}^{(g_2)}(\bar{z}, z_J) \right]. \end{aligned} \quad (5.10)$$

Here

$$K(z_0, z) = \frac{\int_z^{\bar{z}} \omega_2^{(0)}(\cdot, z_0)}{(Y(z) - Y(\bar{z}))dx(z)} \quad (5.11)$$

is called the recursion kernel, and the ' in the last line of (5.8) means that the cases $(g_1, I) = (0, \emptyset)$ and $(g_2, J) = (0, \emptyset)$ must be excluded from the sum.

The Eynard-Orantin differentials $\omega_n^{(g)}$'s are meromorphic multi-differentials on \mathcal{S}^n and are known to be holomorphic except at the branchpoints if $(g, n) \neq (0, 1), (0, 2)$. In [16], the authors also introduced symplectic invariants $F^{(g)}$ defined by

Definition 5.4 (Definition 4.3 of [16]) The g^{th} symplectic invariant of the spectral curve is defined by

$$F^{(2g)} = \frac{1}{2-2g} \sum_{r \text{ branchpoints}} \operatorname{Res}_{z \rightarrow r} \Phi(z) \omega_1^{(g)}(z) \quad \text{for } g \geq 2 \quad (5.12)$$

where

$$\Phi(z) = \int_{z_o}^z Y(\tilde{z})dx(\tilde{z}) \quad (z_o \text{ is a generic point}). \quad (5.13)$$

Note here that the g^{th} free energy is usually denoted by $F^{(g)}$ or F_g in the literature (See [16]). We chose to use a different labeling since in this paper superscripts correspond to the order in \hbar in the series expansion. $F^{(0)}$ and $F^{(2)}$ are defined with specific formulas (see §4.2.2 and §4.2.3 of [16] with a different sign convention).

Note that this definition extends to the case $n \geq 0$ (with the identification $\omega_0^{(g)} = F^{(2g)}$) the following property satisfied by the Eynard-Orantin differentials:

$$\omega_n^{(g)}(z_1, \dots, z_n) = \frac{1}{2-2g-n} \sum_{r \text{ branchpoints}} \operatorname{Res}_{z \rightarrow r} \Phi(z) \omega_{n+1}^{(g)}(z, z_1, \dots, z_n) \quad \text{for } g \geq 0 \quad (5.14)$$

Note also that the Eynard-Orantin differentials or symplectic invariants do not depend on the choice of parametrization $(x(z), Y(z))$.

Remark 5.5 *Since the spectral curve is invariant under gauge transformations of the form $\tilde{\Psi}(x, t) = U(t, \hbar)\Psi(x, t)$ then all Eynard-Orantin differentials and symplectic invariants are trivially invariant under these gauge transformations. One must be careful with the terminology used in “symplectic invariants”. Indeed as proved in [16] and corrected in [17], the $F^{(2g)}$ ’s defined here (following the definition of [16]) are invariant under symplectic transformations of the spectral curve, i.e. transformations $(\tilde{x}, \tilde{Y}) = (F(x, Y), G(x, Y))$ for which $d\tilde{x} \wedge d\tilde{Y} = dx \wedge dY$ except those involving the change $(x, y) \rightarrow (y, -x)$. In [17] a correction to the present formula was presented to obtain the full symplectic invariance but adding such corrections (that are not trivial in the case of the Painlevé equations since the residues $\text{Res}_{z \rightarrow \pm 1} y dx(z)$ are non-zero) would ruin the matching with the tau-function. Moreover we do not need these specific symplectic transformations to obtain the invariance under admissible gauge transformations. Indeed, in general, the Eynard-Orantin differentials and $F^{(2g)}$ may be affected by symplectic transformations but it happens that they remain invariant (except for $\omega_1^{(0)}(z)$) under simple transformations of the form $(\tilde{x} = x, \tilde{Y} = Y + f(x))$. The last point is obvious because the recursion kernel $K(z_0, z)$ and $\omega_2^{(0)}(z_1, z_2)$ are trivially invariant under such transformations. Consequently we get that **the Eynard-Orantin differentials $\omega_n^{(g)}$ with $(n, g) \neq (1, 0)$ and the symplectic invariants $F^{(2g)}$ are invariant under admissible gauge transformations defined in definition 2.4.***

The case of $\omega_1^{(0)}(z)$ is special. It is invariant under gauge transformations of the form (2.29) but is not invariant under gauge transformations of the form (2.28). Easy computations show that:

$$\tilde{\omega}_1^{(0)}(z) = \omega_1^{(0)}(z) - \sum_{i=1}^r \frac{\nu_i \theta_i}{x(z) - a_i} dx(z) \quad (5.15)$$

This will be in complete correspondence with the upcoming definition of the determinantal formulas.

Surprisingly, from the integrable system point of view, it appears that the notion of spectral curve in its present form may not be as fundamental as expected in the literature since it is not invariant under all admissible gauge transformations. However, one can easily create a gauge invariant object by considering the equivalence class of spectral curves (defined by an algebraic equation $\mathcal{E}(x, Y) = 0$) under the equivalence relation \mathcal{R} :

$$(\mathcal{E}(x, Y) = 0) \mathcal{R} \left(\tilde{\mathcal{E}}(\tilde{x}, \tilde{Y}) = 0 \right) \Leftrightarrow \exists f \text{ such that } \tilde{\mathcal{E}}(x, Y + f(x)) = \mathcal{E}(x, Y)$$

In particular, the notions of Eynard-Orantin differentials (for $(n, g) \neq (1, 0)$) and symplectic invariants associated to an equivalence class are well-defined since they are invariant under such transformations.

6 Determinantal formulas and topological type property

6.1 Determinantal formula

In this section we review the determinantal formulas formalism (developed in [2],[3]) that connect the WKB solution of isomonodromic systems to the topological recursion correlation functions. Let

$$\Psi(x) = \begin{pmatrix} \psi(x) & \phi(x) \\ \tilde{\psi}(x) & \tilde{\phi}(x) \end{pmatrix} \quad (6.1)$$

be the WKB solution of the isomonodromic system defined by our Lax pairs (here we are omitting the t -dependence for simplicity). Determinantal formulas are obtained from the Christoffel-Darboux kernel

$$K(x_1, x_2) = \frac{\psi(x_1)\tilde{\phi}(x_2) - \tilde{\psi}(x_1)\phi(x_2)}{x_1 - x_2} \quad (6.2)$$

with the following definition:

Definition 6.1 (Definition 2.3 of [3]) *The (connected) correlation functions are defined by:*

$$W_1(x) = \frac{\partial\psi}{\partial x}(x)\tilde{\phi}(x) - \frac{\partial\tilde{\psi}}{\partial x}(x)\phi(x), \quad (6.3)$$

$$W_n(x_1, \dots, x_n) = -\frac{\delta_{n,2}}{(x_1 - x_2)^2} + (-1)^{n+1} \sum_{\sigma: n\text{-cycles}} \prod_{i=1}^n K(x_i, x_{\sigma(i)}) \quad \text{for } n \geq 2 \quad (6.4)$$

where σ is a n -cycle.

Under assumption (3.1), the correlation functions W_n admit a formal power series in \hbar whose coefficients are symmetric functions of x_1, \dots, x_n . Note that there exists an alternative expression for the correlation functions in terms of the rank 1 projector:

$$M(x) = \Psi(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi^{-1}(x) = \begin{pmatrix} \psi\tilde{\phi} & -\psi\phi \\ \tilde{\psi}\phi & -\phi\tilde{\psi} \end{pmatrix}. \quad (6.5)$$

It is in fact the canonical projector on the first coordinate taken into the basis defined by $\Psi(x)$. The rank 1 projector satisfies:

$$M^2 = M, \quad \text{Tr} M = 1, \quad \det M = 0. \quad (6.6)$$

We remark that, under a general gauge transformation $\tilde{\Psi}(x, t) = U(x, t)\Psi(x, t)$, M is only affected by a simple change of basis: $\tilde{M}(x, t) = U^{-1}(x, t)M(x, t)U(x, t)$. Moreover, theorem 2.1 of [3] gives an alternative expression for $W_n(x_1, \dots, x_n)$:

$$W_1(x) = -\frac{1}{\hbar} \text{Tr}(\mathcal{D}(x)M(x)), \quad (6.7)$$

$$W_2(x_1, x_2) = \frac{\text{Tr}(M(x_1)M(x_2)) - 1}{(x_1 - x_2)^2}, \quad (6.8)$$

$$\begin{aligned} W_n(x_1, \dots, x_n) &= (-1)^{n+1} \text{Tr} \sum_{\sigma: n\text{-cycles}} \prod_{i=1}^n \frac{M(x_{\sigma(i)})}{x_{\sigma(i)} - x_{\sigma(i+1)}} \\ &= \frac{(-1)^{n+1}}{n} \sum_{\sigma \in S_n} \frac{\text{Tr} M(x_{\sigma(1)}) \dots M(x_{\sigma(n)})}{(x_{\sigma(1)} - x_{\sigma(2)}) \dots (x_{\sigma(n-1)} - x_{\sigma(n)})(x_{\sigma(n)} - x_{\sigma(1)})} \quad \text{for } n \geq 3. \end{aligned} \quad (6.9)$$

Note that the definition of the correlation functions only involves the matrix $\mathcal{D}(x, t)$ but not directly the time dependence. In fact these definitions apply for any 2×2 linear system $\partial_x \Psi(x) = \mathcal{D}(x)\Psi(x)$ even if it does not come from a Lax pair. From the alternative definition and the fact that admissible gauge transformations acts on M only through a change of basis, it is obvious that **the correlation functions W_n (except W_1) are invariant under admissible gauge transformations defined in definition 2.4.** We also mention that as presented in [2] and

[3], the correlation functions satisfy the so-called loop equations (i.e. an infinite set of relations connecting the various functions).

The case of $W_1(x)$ is special. It is invariant under gauge transformations of type (2.29) but not under gauge transformations of type (2.28). For such transformations $\tilde{\Psi}(x, t) = \prod_{i=1}^r (x - a_i)^{\frac{\nu_i \theta_i}{\hbar}}$ we find from the alternative definition that:

$$\tilde{W}_1(x) = W_1(x) - \frac{1}{\hbar} \sum_{i=1}^r \frac{\nu_i \theta_i}{x - a_i} \quad (6.10)$$

In particular, if the system satisfies the Topological Type property (6.13) this implies that:

$$\tilde{W}_1^{(0)}(x) = W_1^{(0)}(x) - \sum_{i=1}^r \frac{\nu_i \theta_i}{x - a_i} \text{ and } \tilde{W}_1^{(g)}(x) = W_1^{(g)}(x) \text{ for } g \geq 1 \quad (6.11)$$

in perfect correspondence with (5.15) and remark 5.5.

6.2 Topological Type property

Following the work of Bergère, Borot and Eynard, we now give the definition of the Topological Type property:

Definition 6.2 (Definition 3.3 of [2], Section 2.5 of [3]) *In the case of a genus 0 spectral curve, the differential system $\partial_x \Psi(x) = \mathcal{D}(x)\Psi(x)$ is said to be of topological type (or (TT), for short) if the correlation functions W_n given in definition (6.3) or (6.7) satisfy the following conditions:*

(1) Existence of a series expansion in \hbar : *The correlation functions admit a series expansion in \hbar of the form:*

$$W_n(x_1, \dots, x_n) = \sum_{g=0}^{\infty} W_n^{(g)}(x_1, \dots, x_n) \hbar^g \quad (6.12)$$

(2) Parity property: $W_n|_{\hbar \mapsto -\hbar} = (-1)^n W_n$ holds for $n \geq 1$. *This is equivalent to say that the previous series expansion is even (resp. odd) when n is even (resp. odd).*

(3) Pole structure: *The functions $W_n^{(g)}(x_1, \dots, x_n)$ only have poles at the branchpoints when $(g, n) \neq (0, 1), (0, 2)$, and $W_2^{(0)}(x_1, x_2)$ has a double pole at $x_1 = x_2$ with no other poles. In fact for genus 0 curves we must have $W_2^{(0)}(x(z_1), x(z_2)) dx(z_1) dx(z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$.*

(4) Leading order: *The leading order of the series expansion of the correlation function W_n is at least of order \hbar^{n-2} .*

In summary, the four conditions are necessary and sufficient to prove that the functions W_n admit a series expansion of the form:

$$W_n(x_1, \dots, x_n) = \sum_{g=0}^{\infty} \hbar^{n-2+2g} W_n^{(g)}(x_1, \dots, x_n) \text{ for } n \geq 1. \quad (6.13)$$

with $W_n^{(g)}(x_1, \dots, x_n)$ regular at the even zeros of the spectral curve.

The main interest of the Topological Type property is that it is a sufficient condition to prove the connection with the topological recursion. Indeed, it is proved in [2] and [3] that:

Theorem 6.3 (Theorem 2.1 of [3], Theorem 3.1 and Corollary 4.2 of [2]) *If the differential system (2.1–2.6) satisfies the Topological Type property, then the functions $W_n^{(g)}(x_1, \dots, x_n)$ appearing in the formal expansion (6.13) of the correlation functions $W_n(x_1, \dots, x_n)$ are identical to the Eynard-Orantin differentials $\omega_n^{(g)}(z_1, \dots, z_n)$ obtained from the topological recursion applied on the spectral curve in the following way:*

$$W_n^{(g)}(x(z_1), \dots, x(z_n)) dx(z_1) \cdots dx(z_n) = \omega_n^{(g)}(z_1, \dots, z_n) \quad \text{for } g \geq 0 \text{ and } n \geq 1, \quad (6.14)$$

where $x(z)$ is the parametrization of the spectral curve. Moreover, when the correlation functions comes from a Lax pair, the \hbar expansion of the isomonodromic τ -function (in the sense of [22]) of the differential system (2.1–2.6) matches with the symplectic invariants generating function $F_J^{(g)}(t)$ obtained from the topological recursion applied to the spectral curve in the following sense:

$$\frac{d}{dt} \tau^{(2g)} = - \frac{dF^{(2g)}}{dt}(t) = \quad \text{for } g \geq 0 \text{ where } \frac{d}{dt} \tau^{(2g)} \text{ is defined in (4.8)} \quad (6.15)$$

Note that the minus sign arising in (6.15) is just a matter of convention regarding the definition of the $F^{(g)}$ (we followed definition of [16]). The last theorem is particularly interesting since it shows that the topological recursion reconstructs the formal series expansions of the determinantal formulas and the tau-function of the Lax system. Note that the Topological Type property is a sufficient condition for a Lax system to be reconstructed from the topological recursion. However at the moment it is not known whether the Topological Type property is really necessary but so far all known cases in which the topological recursion is known to reconstruct the determinantal formulas are of Topological Type, yet it is still unclear if this situation is generic or not.

Remark 6.4 • *The Topological Type property is invariant under admissible gauge transformations presented in definition 2.4. Indeed, we have seen earlier that for $n \geq 2$ the correlation functions $W_n(x_1, \dots, x_n)$ are invariant under admissible gauge transformations ($W_1(x)$ being studied separately in (6.10)) so that if they satisfy the Topological Type property in a gauge, they will satisfy the same property in any other gauge connected to the previous one by an admissible gauge transformation.*

- *The sign in (6.15) is a matter of conventions chosen in [16]. Since we kept the notations of [16] for the definition of the symplectic invariants we get an overall minus sign.*

6.3 Main theorem

Our main theorem is formulated as follows:

Theorem 6.5 *In all six Painlevé cases with generic monodromy parameters θ_* satisfying Assumption 3.3, the Lax pairs (2.1–2.6) associated with the Painlevé equations are of Topological Type. Therefore, the \hbar -expansion of the τ -function and correlation functions W_n are respectively identified with the generating functions of symplectic invariants $F_J^{(g)}$ and the correlation functions $\omega_{J,n}^{(g)}$*

computed from the topological recursion applied to the corresponding spectral curve as follows:

$$\frac{1}{\hbar^2} \frac{d}{dt} \ln \tau(t, \hbar) = -\frac{d}{dt} \sum_{g=0}^{\infty} \hbar^{2g-2} F^{(2g)}(t), \quad (6.16)$$

$$W_n(x(z_1), \dots, x(z_n)) dx(z_1) \cdots dx(z_n) = \sum_{g=0}^{\infty} \hbar^{2g-2+n} \omega_n^{(g)}(z_1, \dots, z_n) \quad \text{for } n \geq 1. \quad (6.17)$$

To support and complete our theorem we provide in appendix I the specific computations of $F^{(0)}$ and $F^{(1)}$ and prove that they match with $-\frac{d}{dt}\tau^{(0)}$ and $-\frac{d}{dt}\tau^{(2)}$ up to constants (the tau-function being defined up to constants, only the matching of the derivatives makes real sense). The general proof of the Topological Type property is provided in Appendices E, F, G, H where we will prove that the four conditions (1) \sim (4) hold for all Lax pairs listed in section 2.1. The idea of our proof is that **choosing a gauge in which all quantities, including $\mathcal{D}(x, t)$, $\mathcal{R}(x, t)$ and $M(x, t)$ have nice \hbar series expansion will allow us to show that the Topological Type property is verified in this gauge**. Since the correlation functions, the Eynard-Orantin differentials and the Topological Type property are invariant under any admissible gauge transformations we can **extend this result for any Lax pairs connected to ours with an admissible gauge transformation** (thus including Jimbo-Miwa Lax pairs). In order to prove the four conditions defining the Topological Type property, and especially the pole structure condition, we use to our advantage the additional time differential equations that characterize our Lax pairs. This shows the important role played by the isomonodromic time t in the Painlevé equations.

7 Conclusion

In this article we showed how to introduce by rescaling of the monodromy parameters a small formal \hbar parameter in the formalism of the six Painlevé Lax pairs. In this formalism we then presented a proof of the Topological Type property for all six Painlevé cases as well as the various connections with the tau-function, Okamoto's σ -functions and the Hamiltonians of the underlying problems. The proof of the Topological Type property implies that for the six Painlevé equations, we can match the \hbar -formal series expansions of the correlation functions (i.e. determinantal formulas) W_n and the tau-functions $\ln \tau$ with respectively the Eynard-Orantin differentials $\omega_n^{(g)}$ and symplectic invariants $F^{(2g)}$ computed from the topological recursion applied on the spectral curve associated to the Lax pair. We also discussed the validity of this result under certain gauge transformations of the Lax pairs. Several questions arise from this work that would deserve further study:

- We have introduced the \hbar parameter with a rescaling of the monodromy parameters but it would be interesting to see if other interesting rescalings lead to regimes for which a spectral curve can be defined.
- The approach developed in this article is purely formal since we assumed that solutions of Painlevé equations had a series expansion in \hbar . However it is well known in matrix models and in perturbation theory that these series expansions may not be convergent but only Borel summable at $\hbar = 0$. Therefore a natural extension to our work would be to study the convergence of the series introduced in this paper. In particular, the fact that our spectral curves are of genus 0 allows to hope that the series may be convergent at least in an open neighborhood of $\hbar = 0$.

- Connections between the topological recursion, matrix models, and integrable systems are now well documented in the literature and this article corroborates this point. However, since the strategy presented in this article (selection of a good gauge, introduction of \hbar , computation of the spectral curve and proof of the topological type property) is quite general, it seems that it could be tried successfully on other integrable systems.
- On the integrable system side, our analysis shows that the important quantity is the matrix $M(x, t)$ rather than the matrix $\Psi(x, t)$ whose gauge freedom artificially hardens the discussion (in a similar way as the Ricatti versus Schrödinger equation). In fact, we believe that all interesting physical quantities should be invariant under admissible gauge transformations thus giving some legitimacy to the definition of correlation functions. In particular, it would be interesting to redefine the initial Lax pair problem exclusively in terms of the matrix $M(x, t)$.
- In this article we introduced the \hbar parameter from the Lax pair (rescaling of the parameters) and deduced the corresponding modifications on the Hamiltonians underlying the Painlevé equations. Surprisingly we found that for Painlevé equations 1, 2, 4 and 5, the introduction of the \hbar parameter does not change the explicit expression of the Hamiltonians (see theorem 2.6). For Painlevé equations 3 and 6 only a linear term in \hbar appears and we can always obtain the tau-function (see (2.51)) by taking $\ln \tau_J(t) = H_J(p(t), q(t), t, \hbar = 0)$. This suggests that our parameter \hbar may have some nice interpretation in the Hamiltonian formulation.

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A Connection with Jimbo-Miwa Lax pairs

In [23], the authors produced a list of Lax pairs corresponding to all six Painlevé equations. In this paper we defined slightly different Lax pairs and for completeness we describe here the various transformations connecting both sets. To avoid confusion, we denote with an index (JM) all quantities appearing in Jimbo-Miwa paper [23]. The Jimbo-Miwa Lax pairs are:

$$\frac{\partial}{\partial x} Y_{JM}(x, t) = A_{JM}(x, t) Y_{JM}(x, t) \text{ and } \frac{\partial}{\partial t} Y_{JM}(x, t) = B_{JM}(x, t) Y_{JM}(x, t)$$

where $A_{JM}(x, t)$ and $B_{JM}(x, t)$ are 2×2 matrices.

- The Lax pair for Painlevé I is the same as our Lax pair (2.1) with the identification $(p, q) = (z_{JM}, y_{JM})$.
- The Lax pair for Painlevé II proposed by Jimbo and Miwa is:

$$\begin{aligned} A_{JM}(x, t) &= \begin{pmatrix} x^2 + z_{JM} + \frac{t}{2} & u_{JM}(x - y_{JM}) \\ -2u_{JM}^{-1}(z_{JM}x + z_{JM}y_{JM} + \theta) & -(x^2 + z_{JM} + \frac{t}{2}) \end{pmatrix} \\ B_{JM}(x, t) &= \frac{1}{2} \begin{pmatrix} x & u \\ -2u_{JM}^{-1}z_{JM} & -x \end{pmatrix}. \end{aligned}$$

where two t -dependent functions $z_{JM}(t)$ and $u_{JM}(t)$ are introduced. Our Lax pair (2.2) is connected to the former one with:

$$\Psi(x, t) = \begin{pmatrix} u_{JM}^{-\frac{1}{2}}(t) & 0 \\ 0 & u_{JM}^{\frac{1}{2}}(t) \end{pmatrix} Y_{JM}(x, t) \text{ and } (p, q) = (z_{JM}, y_{JM}).$$

- The Lax pair for Painlevé III proposed by Jimbo and Miwa is given by:

$$\begin{aligned} A_{JM}(x, t) &= \frac{t}{2}\sigma_3 + \frac{1}{x} \begin{pmatrix} -\frac{\theta_\infty}{2} & u_{JM} \\ v_{JM} & \frac{\theta_\infty}{2} \end{pmatrix} + \frac{1}{x^2} \begin{pmatrix} z_{JM} - \frac{t}{2} & -w_{JM}z_{JM} \\ w_{JM}^{-1}(z_{JM} - t) & -z_{JM} + \frac{t}{2} \end{pmatrix} \\ B_{JM}(x, t) &= \frac{x}{2}\sigma_3 + \frac{1}{t} \begin{pmatrix} 0 & u_{JM} \\ v_{JM} & 0 \end{pmatrix} - \frac{1}{2tx} \begin{pmatrix} z_{JM} - \frac{t}{2} & -w_{JM}z_{JM} \\ w_{JM}^{-1}(z_{JM} - t) & -z_{JM} + \frac{t}{2} \end{pmatrix} \end{aligned}$$

where $u_{JM} = -z_{JM}y_{JM}w_{JM}$ and $v_{JM} = \frac{-(z_{JM}-t)y_{JM}-\theta_\infty+\frac{t}{2z_{JM}}(\theta_0+\theta_\infty)}{w_{JM}}$. The connection with our Lax pair (2.3) is given by the gauge transformation:

$$\Psi(x, t) = \begin{pmatrix} w_{JM}^{-\frac{1}{2}}(t) & 0 \\ 0 & w_{JM}^{\frac{1}{2}}(t) \end{pmatrix} Y_{JM}(x, t) \text{ and } (p, q) = (z_{JM}, y_{JM})$$

- The Lax pair for Painlevé IV proposed by Jimbo and Miwa is given by:

$$\begin{aligned} A_{JM}(x, t) &= \begin{pmatrix} x + t - \frac{z_{JM}-\theta_0}{x} & u_{JM} \left(1 - \frac{y_{JM}}{2x}\right) \\ \frac{2}{u_{JM}}(z_{JM} - \theta_0 - \theta_\infty) + \frac{2z_{JM}(z_{JM}-2\theta_0)}{u_{JM}y_{JM}x} & -x - t + \frac{z_{JM}-\theta_0}{x} \end{pmatrix} \\ B_{JM}(x, t) &= \begin{pmatrix} x & u_{JM} \\ \frac{2}{u_{JM}}(z_{JM} - \theta_0 - \theta_\infty) & -x \end{pmatrix} \end{aligned}$$

Our Lax pair (2.4) can be obtained from the former Lax pair by the gauge transformation:

$$\Psi(x, t) = \begin{pmatrix} u_{JM}^{-\frac{1}{2}} & 0 \\ 0 & u_{JM}^{\frac{1}{2}} \end{pmatrix} Y_{JM}(x, t)$$

and the identification $(p, q) = \left(-\frac{2z_{JM}}{y_{JM}}, \frac{y_{JM}}{2}\right)$.

- The Lax pair for Painlevé V proposed by Jimbo and Miwa is:

$$\begin{aligned} A_{JM}(x, t) &= \begin{pmatrix} \frac{t}{2} + \frac{z_{JM} + \frac{\theta_0}{2}}{x} - \frac{z_{JM} + \frac{\theta_0 + \theta_\infty}{2}}{x-1} & -\frac{u_{JM}(z_{JM} + \theta_0)}{x} + \frac{u_{JM}y_{JM}(z_{JM} + \frac{\theta_0 - \theta_1 + \theta_\infty}{2})}{x-1} \\ \frac{z_{JM}}{u_{JM}x} - \frac{z_{JM} + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}}{u_{JM}y_{JM}(x-1)} & -\frac{t}{2} - \frac{z_{JM} + \frac{\theta_0}{2}}{x} + \frac{z_{JM} + \frac{\theta_0 + \theta_\infty}{2}}{x-1} \end{pmatrix} \\ B_{JM}(x, t) &= \begin{pmatrix} \frac{x}{2} & \frac{u_{JM}}{t}(z_{JM} + \theta_0 - y_{JM}(z_{JM} + \frac{\theta_0 - \theta_1 + \theta_\infty}{2})) \\ \frac{1}{tu_{JM}}(z_{JM} - \frac{1}{y_{JM}}(z_{JM} + \frac{\theta_0 + \theta_1 + \theta_\infty}{2})) & -\frac{x}{2} \end{pmatrix} \end{aligned}$$

We note here two typos in the Lax pair proposed in [23] where $B_{JM}(x, t)$ lacks the x factor and the sign of the $(1, 2)$ entry is not correct. We can obtain the Lax pair proposed in (2.5) from this one by the gauge transformation:

$$\Psi(x, t) = \begin{pmatrix} u_{JM}^{-\frac{1}{2}} & 0 \\ 0 & u_{JM}^{\frac{1}{2}} \end{pmatrix} Y_{JM}(x, t)$$

and the identification $(p, q) = (z_{JM}y_{JM}, \frac{1}{y_{JM}})$.

- The Lax pair for Painlevé VI proposed by Jimbo and Miwa is:

$$A_{JM}(x, t) = \frac{(A_0)_{JM}}{x} + \frac{(A_1)_{JM}}{x-1} + \frac{(A_t)_{JM}}{x-t} \text{ and } B(x, t) = -\frac{(A_t)_{JM}}{x-t}$$

where the matrices $(A_0)_{JM}$, $(A_1)_{JM}$ and $(A_t)_{JM}$ are defined by:

$$\begin{aligned} (A_0)_{JM} &= \begin{pmatrix} (z_0)_{JM} + \theta_0 & -u_{JM}(z_0)_{JM} \\ u_{JM}^{-1}((z_0)_{JM} + \theta_0) & -(z_0)_{JM} \end{pmatrix} \\ (A_1)_{JM} &= \begin{pmatrix} (z_1)_{JM} + \theta_1 & -v_{JM}(z_1)_{JM} \\ v_{JM}^{-1}((z_1)_{JM} + \theta_1) & -(z_1)_{JM} \end{pmatrix} \\ (A_t)_{JM} &= \begin{pmatrix} (z_t)_{JM} + \theta_t & -w_{JM}(z_t)_{JM} \\ w_{JM}^{-1}((z_t)_{JM} + \theta_t) & -(z_t)_{JM} \end{pmatrix} \\ (A_\infty)_{JM} &= \begin{pmatrix} \frac{1}{2}(\theta_\infty - \theta_0 - \theta_1 - \theta_t) & 0 \\ 0 & -\frac{1}{2}(\theta_\infty + \theta_0 + \theta_1 + \theta_t) \end{pmatrix} = -(A_0 + A_1 + A_t) \end{aligned}$$

with:

$$u_{JM} = \frac{k_{JM}y_{JM}}{t(z_0)_{JM}}, \quad v_{JM} = -\frac{k_{JM}(y_{JM} - 1)}{(t-1)(z_1)_{JM}}, \quad w_{JM} = \frac{k_{JM}(y_{JM} - t)}{t(t-1)(z_t)_{JM}}$$

Our Lax pair (2.6) is connected to the previous Lax pair via a gauge transformation:

$$\Psi(x, t) = x^{-\frac{\theta_0}{2}}(x-1)^{-\frac{\theta_1}{2}}(x-t)^{-\frac{\theta_t}{2}} \begin{pmatrix} k_{JM}(t)^{-\frac{1}{2}} & 0 \\ 0 & k_{JM}(t)^{\frac{1}{2}} \end{pmatrix} Y_{JM}(x, t)$$

and the identifications:

$$z_0 = (z_0)_{JM}, \quad z_1 = (z_1)_{JM}, \quad z_t = (z_t)_{JM} \text{ and } (p, q) = (z_{JM}, y_{JM}).$$

B Derivation of the \hbar -deformed Painlevé equations from the Lax pairs

The introduction of the \hbar parameter in the Lax pair after rescaling modifies to some extent the various equations obtained from the compatibility equation

$$\hbar \partial_t \mathcal{D}(x, t) - \hbar \partial_x \mathcal{R}(x, t) + [\mathcal{D}(x, t), \mathcal{R}(x, t)] = 0.$$

In this section we present those modifications as well as their consequences on the final Painlevé equations.

B.1 Painlevé I

The compatibility equation for (2.1) gives the following system of equations:

$$\hbar \dot{p} = 6q^2 + t \text{ and } \hbar \dot{q} = p, \quad (\text{B.1})$$

from which it is trivial to deduce that $q(t)$ satisfy the \hbar -modified Painlevé 1 equation:

$$\hbar^2 \ddot{q} = 6q^2 + t.$$

B.2 Painlevé II

The compatibility equation for (2.2) gives the following system of equations:

$$\hbar \dot{p} = -2qp - \theta \text{ and } \hbar \dot{q} = p + q^2 + \frac{t}{2}. \quad (\text{B.2})$$

Taking the derivative of the first equation and inserting it back into the second equation gives that $q(t)$ satisfies the \hbar -modified Painlevé 2 equation:

$$\hbar^2 \ddot{q} = 2q^3 + tq + \frac{\hbar}{2} - \theta.$$

B.3 Painlevé III

The compatibility equation for (2.3) gives the following system of equations:

$$\hbar \dot{p} = \frac{1}{t} [-4qp^2 - p(-4tq + 2\theta_\infty - \hbar) + t(\theta_0 + \theta_\infty)] \text{ and } \hbar \dot{q} = \frac{1}{t} [4q^2p - 2tq^2 + q(2\theta_\infty - \hbar) + 2t]. \quad (\text{B.3})$$

Extracting $p(t)$ from the second equation and inserting it back into the first one gives that $q(t)$ satisfies the \hbar -modified Painlevé 3 equation:

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{q} \dot{q}^2 - \frac{\hbar^2}{t} \dot{q} + \frac{4}{t} (\theta_0 q^2 - \theta_\infty + \hbar) + 4q^3 - \frac{4}{q}.$$

B.4 Painlevé IV

The compatibility equation for (2.4) gives the following system of equations:

$$\hbar \dot{p} = -p^2 - 4pq - 2tp - 2(\theta_0 + \theta_\infty) \text{ and } \hbar \dot{q} = 2(pq + q^2 + tq + \theta_0). \quad (\text{B.4})$$

Extracting p from the second equation and inserting it back into the first equation gives (after some substantial computations) that $q(t)$ satisfies the \hbar -modified Painlevé 4 equation:

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{2q} \dot{q}^2 + 2 \left(3q^3 + 4tq^2 + (t^2 - 2\theta_\infty + \hbar)q - \frac{\theta_0^2}{q} \right).$$

B.5 Painlevé V

The compatibility equation for (2.5) gives the following system of equations:

$$\begin{aligned} t\hbar \frac{d(pq)}{dt} &= -q(q^2 - 1)p^2 + \left(-q^2 \frac{3\theta_0 + \theta_1 + \theta_\infty}{2} + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) p - \frac{q\theta_0(\theta_0 + \theta_1 + \theta_\infty)}{2} \\ 2t\hbar \dot{q}(2pq + \theta_0 - \theta_1 + \theta_\infty) - 4t\hbar q \frac{d(pq)}{dt} &= 4p^2 q^2 (3q - 1)(q - 1) \\ &+ 4pq \left((4\theta_0 + 2\theta_\infty)q^2 - q(t + 4\theta_0 + 2\theta_1 + 3\theta_\infty) + \theta_0 - \theta_1 + \theta_\infty \right) \\ &+ q^2 (5\theta_0^2 - \theta_1^2 + \theta_\infty^2 + 6\theta_0\theta_\infty) - 2q(\theta_0 - \theta_1 + \theta_\infty)(t + 2\theta_0 + \theta_\infty) + (\theta_0 - \theta_1 + \theta_\infty)^2. \end{aligned}$$

This is equivalent to:

$$\begin{aligned} t\hbar \dot{q} &= 2q(q - 1)^2 p + \frac{3\theta_0 + \theta_1 + \theta_\infty}{2} q^2 - (t + 2\theta_0 + \theta_\infty)q + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \\ t\hbar \dot{p} &= -(3q^2 - 4q + 1)p^2 + (-(3\theta_0 + \theta_1 + \theta_\infty)q + t + 2\theta_0 + \theta_\infty)p - \frac{1}{2}\theta_0(\theta_0 + \theta_1 + \theta_\infty). \end{aligned} \quad (\text{B.5})$$

Extracting p from the first equation and inserting it back into the second gives that $q(t)$ satisfies the \hbar -modified Painlevé 5 equation:

$$\begin{aligned} \hbar^2 \ddot{q} &= \left(\frac{1}{2q} + \frac{1}{q - 1} \right) (\hbar \dot{q})^2 - \hbar^2 \frac{\dot{q}}{t} + \frac{(q - 1)^2}{t^2} \left(\alpha q + \frac{\beta}{q} \right) + \frac{\gamma q}{t} + \frac{\delta q(q + 1)}{q - 1} \\ \text{with } \alpha &= \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{8}, \beta = -\frac{(\theta_0 - \theta_1 + \theta_\infty)^2}{8}, \gamma = \theta_0 + \theta_1 - \hbar \text{ and } \delta = -\frac{1}{2}. \end{aligned}$$

B.6 Painlevé VI

Following the various steps proposed in [23], one can follow the introduction of the \hbar parameter in the Painlevé 6 system. We find:

$$\begin{aligned} \hbar t(t - 1)\dot{q} &= 2q(q - 1)(q - t)p - \theta_0(q - 1)(q - t) - \theta_1 q(q - t) - (\theta_t - \hbar)q(q - 1) \\ \hbar t(t - 1)\dot{p} &= (-3q^2 + 2q(t + 1) - t)p^2 + ((2q - t - 1)\theta_0 + (2q - t)\theta_1 + (2q - 1)(\theta_t - \hbar))p \\ &\quad - \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)(\theta_0 + \theta_1 + \theta_t + \theta_\infty - 2\hbar) \end{aligned} \quad (\text{B.6})$$

Extracting p from the first equation and inserting it back into the first one gives that q satisfies the \hbar -modified Painlevé 6 equation:

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{2} \left(\frac{1}{q} + \frac{1}{q - 1} + \frac{1}{q - t} \right) \dot{q}^2 - \hbar^2 \left(\frac{1}{t} + \frac{1}{t - 1} + \frac{1}{q - t} \right) \dot{q}$$

$$+\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{q^2}+\gamma\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right] \quad (\text{B.7})$$

where the parameters are:

$$\alpha = \frac{1}{2}(\theta_\infty - \hbar)^2, \beta = -\frac{\theta_0^2}{2}, \gamma = \frac{\theta_1^2}{2} \text{ and } \delta = \frac{\hbar^2 - \theta_t^2}{2}.$$

C Algebraic equations satisfied by $q_0(t)$

Considering the leading order in \hbar , we have that $q_0(t)$ satisfies an algebraic equation. These equations can easily be obtained from the \hbar -deformed versions of the Painlevé equations after discarding all higher \hbar terms. Equivalently, they can also be obtained from the Hamiltonian formalism by requiring that $\frac{\partial H_J}{\partial p} = \frac{\partial H_J}{\partial q} = 0$. We note these algebraic relation by $E_J(q_0, t) = 0$ with $J \in \{1, \dots, 6\}$ where E_J is a polynomial. We list here these equations in the case of our Lax pairs:

- Painlevé I:

$$6q_0^2 + t = 0 \Leftrightarrow q_0(t) = \pm\sqrt{\frac{t}{6}} \quad (\text{C.1})$$

- Painlevé II:

$$2q_0^3 + tq_0 - \theta = 0 \quad (\text{C.2})$$

- Painlevé III:

$$tq_0^4 + \theta_0q_0^3 - \theta_\infty q_0 - t = 0 \quad (\text{C.3})$$

- Painlevé IV:

$$3q_0^4 + 4tq_0^3 + (t^2 - 2\theta_\infty)q_0^2 - \theta_0^2 = 0 \quad (\text{C.4})$$

- Painlevé V:

$$\frac{(q_0 - 1)^2}{t^2} \left(\frac{q_0(\theta_0 - \theta_1 + \theta_\infty)^2}{8} - \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{8q_0} \right) + (\theta_0 + \theta_1)\frac{q_0}{t} - \frac{1}{2}\frac{q_0(q_0 + 1)}{q_0 - 1} = 0 \quad (\text{C.5})$$

which is equivalent to a polynomial equation of degree 5:

$$0 = q_0^5 - 3q_0^4 - 2q_0^3 \left(-1 + \frac{2t^2 - 4t(\theta_0 + \theta_1) + 2\theta_\infty(\theta_0 - \theta_1)}{(\theta_0 - \theta_1 - \theta_\infty)^2} \right) - 2q_0^2 \left(-1 + \frac{2t^2 + 4t(\theta_0 + \theta_1) - 6\theta_\infty(\theta_0 - \theta_1)}{(\theta_0 - \theta_1 - \theta_\infty)^2} \right) - 3 \left(\frac{\theta_0 - \theta_1 + \theta_\infty}{\theta_0 - \theta_1 - \theta_\infty} \right)^2 q_0 + \left(\frac{\theta_0 - \theta_1 + \theta_\infty}{\theta_0 - \theta_1 - \theta_\infty} \right)^2 \quad (\text{C.6})$$

- Painlevé VI:

$$\theta_\infty^2 - \frac{\theta_0^2 t}{q_0^2} + \frac{(t-1)\theta_1^2}{(q_0-1)^2} - \frac{\theta_t^2 t(t-1)}{(q_0-t)^2} = 0 \quad (\text{C.7})$$

This is equivalent to a polynomial equation of degree 6:

$$\begin{aligned} 0 = & \theta_\infty^2 q_0^6 - 2(t+1)\theta_\infty^2 q_0^5 + (-t\theta_0^2 + (t-1)\theta_1^2 + (t^2 + 4t + 1)\theta_\infty^2 - t(t-1)\theta_t^2) q_0^4 \\ & + 2t((t+1)\theta_0^2 - (t-1)\theta_1^2 - (t+1)\theta_\infty^2 + (t-1)\theta_t^2) q_0^3 \\ & - t((t^2 + 4t + 1)\theta_0^2 - t(t-1)\theta_1^2 - t\theta_\infty^2 + (t-1)\theta_t^2) q_0^2 \\ & + 2t^2(t+1)\theta_0^2 q_0 - t^3\theta_0^2 \end{aligned} \quad (\text{C.8})$$

D Spectral curve for Painlevé V and Painlevé VI

In this appendix, we present the computations required to obtain the various results regarding the spectral curves of the Painlevé 5 and Painlevé 6 case.

D.1 Spectral curve for Painlevé V

Projecting the compatibility equations and the Painlevé V equation gives that p_0 and q_0 must obey:

$$\begin{aligned} 0 &= p_0 \left(p_0 + \frac{\theta_0 - \theta_1 + \theta_\infty}{2q_0} \right) - (p_0 q_0 + \theta_0) \left(p_0 q_0 + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) \\ 0 &= t - 2p_0(q_0 - 1)^2 + (q_0 - 1) \left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2q_0} - \frac{3\theta_0 + \theta_1 + \theta_\infty}{2} \right) \end{aligned} \quad (\text{D.1})$$

The determinant of $\mathcal{D}(x, t)$ is given by:

$$\begin{aligned} \Delta &= \frac{t^2}{4x^2(x-1)^2} \left(x(x-1) - \frac{\theta_\infty}{t}x - \frac{2p_0 q_0 + \theta_0}{t} \right)^2 \\ &\quad + \frac{1}{x^2(x-1)^2} \left(-(x-1)(p_0 q_0 + \theta_0) + x \left(p_0 + \frac{\theta_0 - \theta_1 + \theta_\infty}{2q_0} \right) \right) \left(p_0 q_0(x-1) - q_0 x \left(p_0 q_0 + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) \right) \end{aligned} \quad (\text{D.2})$$

From the first equation of (D.1) it is easy to see that the second term of (D.2) admits a double zero $Q_0(t)$ verifying:

$$Q_0 = -\frac{p_0}{p_0(q_0 - 1) + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}} = \frac{p_0 q_0 + \theta_0}{p_0 q_0 + \theta_0 - \left(p_0 + \frac{\theta_0 - \theta_1 + \theta_\infty}{2q_0} \right)} \quad (\text{D.3})$$

The second equation of (D.1) shows that Q_0 is also a zero of $x(x-1) - \frac{\theta_\infty}{t}x - \frac{2p_0 q_0 + \theta_0}{t}$. Consequently we also get that:

$$Q_0 = \frac{1}{2} \left(1 + \frac{\theta_\infty}{t} \right) + \frac{1}{2} \sqrt{\left(1 + \frac{\theta_\infty}{t} \right)^2 - \frac{4(2p_0 q_0 + \theta_0)}{t}} \quad (\text{D.4})$$

Solving for p_0 in the second equation of (D.1) shows from (D.3) that we also have:

$$\begin{aligned} Q_0 &= \frac{-(3\theta_0 + \theta_1 + \theta_\infty)(q_0 - 1)^2 + 2(t - \theta_0 - \theta_1)(q_0 - 1) + 2t}{(q_0 - 1)((\theta_0 - \theta_1 - \theta_\infty)(q_0 - 1)^2 - 2(t + \theta_\infty)(q_0 - 1) - 2t)} \\ &= \frac{q_0((\theta_0 - \theta_1 - \theta_\infty)(q_0 - 1)^2 + 2(t - \theta_0 - \theta_1)(q_0 - 1) + 2t)}{(q_0 - 1)((\theta_0 - \theta_1 - \theta_\infty)(q_0 - 1)^2 + 2(t - \theta_\infty)(q_0 - 1) + 2t)} \end{aligned} \quad (\text{D.5})$$

The last two expressions are equivalent since q_0 satisfies (C.5). Eventually the determinant reads:

$$\Delta = -\frac{t^2(x - Q_0)^2(x^2 - Sx + P)}{4x^2(x-1)^2} \quad (\text{D.6})$$

where at the moment S and P are the remaining unknown parameters of Δ . In fact they are given by:

$$S = \frac{1}{2tq_0} [-(\theta_0 - \theta_1 - \theta_\infty)q_0^2 + 2(\theta_\infty + t)q_0 + \theta_0 - \theta_1 + \theta_\infty]$$

$$P = \frac{-1}{16t^2q_0^2(q_0-1)^2} [(\theta_0 - \theta_1 - \theta_\infty)q_0^3 - (3\theta_0 - 3\theta_1 - \theta_\infty + 2t)q_0^2 + (3\theta_0 - 3\theta_1 + \theta_\infty - 2t)q_0 - \theta_0 + \theta_1 - \theta_\infty] \quad (\text{D.7})$$

Another way to obtain the coefficients (Q_0, S, P) directly is to look at the asymptotic at $x \sim 0$, $x \sim 1$ and $x \sim \infty$. Using the form of the matrix $\mathcal{D}(x, t)$ we have that $\det \mathcal{D} \underset{x \rightarrow 0}{\sim} -\frac{\theta_0^2}{4x^2}$, $\det \mathcal{D}(x, t) \underset{x \rightarrow 1}{\sim} -\frac{\theta_1^2}{4(x-1)^2}$ and $\det \mathcal{D}(x, t) \underset{x \rightarrow \infty}{=} -\frac{t^2}{4} + \frac{t\theta_\infty}{2x} + O\left(\frac{1}{x^2}\right)$. Therefore the parameters (Q_0, S, P) of the spectral curve are characterized by the following system of equations:

$$\begin{aligned} t^2 Q_0^2 P &= \theta_0^2 \\ t^2 (1 - Q_0)^2 (1 - S + P) &= \theta_1^2 \\ t(2Q_0 - 2 + S) &= 2\theta_\infty \end{aligned} \quad (\text{D.8})$$

This system can be solved and we get that the double zero Q_0 of $\Delta(x, t)$ must satisfy the algebraic equation:

$$\begin{aligned} 0 &= 2t^2 Q_0^5 - t(5t + 2\theta_\infty)Q_0^4 + 4t(t + \theta_\infty)Q_0^3 - ((t + \theta_\infty)^2 - (\theta_0^2 - \theta_1^2 + \theta_\infty^2))Q_0^2 - 2\theta_0^2 Q_0 + \theta_0^2 \\ 0 &= Q_0^2(Q_0 - 1)^2(2Q_0 - 1)t^2 - 2\theta_\infty Q_0^2(Q_0 - 1)^2 t + (Q_0(\theta_0 + \theta_1) - \theta_0)(Q_0(\theta_0 - \theta_1) - \theta_0) \end{aligned} \quad (\text{D.9})$$

This provides a direct evolution of the double zero $Q_0(t)$ in terms of t . Note that this evolution is completely independent of q_0 and p_0 and only depends on the monodromy parameters. We now need to rule out possible non-generic cases:

- The double zero may never equal 0 or 1. Indeed in that case (D.9) implies that $\theta_0^2 = 0$ and $-\theta_1^2 = 0$ respectively.
- The simple zeros may never equal 0 or 1. Indeed in that case (D.8) implies that $\theta_0 = 0$ or $\theta_1 = 0$ respectively.
- The simple zeros may never coincide. Indeed in that case we would get $P = d^2$ and $S = 2d$ in (D.8). Solving the first and third equations leads to $Q_0 + d = 1 + \frac{\theta_\infty}{t}$ and $dQ_0 = \frac{\epsilon_0 \theta_0}{t}$. Inserting these relations into the second equation $t^2(Q_0 - 1)^2(d - 1)^2 = \theta_1^2$ is equivalent to $\frac{1}{t} \prod_{(\epsilon_0, \epsilon_1) \in \{\pm 1\}^2} (\theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1) = 0$. Since $t \neq 0$, we get that the simple zeros never coincide as soon as:

$$\theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 \neq 0 \text{ for all choices of } (\epsilon_0, \epsilon_1) \in \{\pm 1\}^2 \quad (\text{D.10})$$

These conditions are exactly the same as those requiring that Painlevé 5 is non-degenerate.

- Eventually, we need to rule out the possibility that one of the simple zero becomes equal to the double zero Q_0 . Let us observe that we have:

$$Q_0 = -\frac{(3\theta_0 + \theta_1 + \theta_\infty)q_0^2 - 2(2\theta_0 + \theta_\infty + t)q_0 + \theta_0 - \theta_1 + \theta_\infty}{(\theta_0 - \theta_1 - \theta_\infty)q_0^2 - 2(\theta_0 - \theta_1 + t)q_0 + \theta_0 - \theta_1 + \theta_\infty}$$

Moreover assuming that Q_0 is a triple zero of $\Delta(x)$ is equivalent to saying that $\Delta''(Q_0) = 0$. This provides a polynomial relation $P(q_0, t)$ between q_0 and t . This polynomial is originally of degree 4 in t and 8 in q_0 . However, since q_0 and t are related by a polynomial relation $E_5(q_0, t) = 0$ (Cf. (C.5)) which is of degree 2 in t , requiring $P(q_0, t) = 0$ is equivalent to require that $\gcd(P, E_5)(q_0, t) = 0$ therefore leading to a new polynomial relation $R(q_0, t) = 0$

of degree 1 in t . Solving this equation in t is easy and let us denote $t = g(q_0)$ the corresponding solution where $z \mapsto g(z)$ is a rational function of z whose expression is explicit (but lengthy so we omit it here). Hence so far we have proved that the spectral curve admits a triple zero if and only if $t = g(q_0)$ with g an explicit rational function. We can then insert $t = g(q_0)$ into (C.5) and we get that:

$$4(q_0^2 - 1)^3 q_0^2 ((\theta_0 + \theta_1 + \theta_\infty)q_0 + \theta_0 - \theta_1 + \theta_\infty)^2 R_9(q_0) = 0$$

where $R_9(z)$ is given by (3.10). It is obvious to see that $q_0 = -\frac{\theta_0 - \theta_1 + \theta_\infty}{\theta_0 + \theta_1 + \theta_\infty}$ does not lead to an admissible solution since it corresponds to $t = 0$. Consequently we recognize here that Q_0 is a triple zero implies that t is a singular time. The converse can also easily be verified so that Q_0 is a triple zero if and only if t is a singular time.

D.2 Spectral curve for Painlevé VI

The computation of the spectral curve for Painlevé VI leads to:

$$Y^2 = \frac{\theta_\infty^2 (x - q_0)^2 P_2(x)}{4(x - t)^2 x^2 (x - 1)^2} \text{ with } P_2(x) = x^2 + \left(-1 - \frac{\tilde{\theta}_0^2 t^2}{q_0^2} + \frac{\tilde{\theta}_1^2 (t - 1)^2}{(q_0 - 1)^2} \right) x + \frac{\tilde{\theta}_0^2 t^2}{q_0^2}. \quad (\text{D.11})$$

Note that we can factorize θ_∞^2 and rewrite the spectral curve only in terms of reduced monodromy parameters defined by $\tilde{\theta}_i = \frac{\theta_i}{\theta_\infty}$ for $i \in \{0, 1, t\}$. Moreover $P_2(x)$ also admits a more symmetric formulation. Using (C.7) we can reformulate it into:

$$Y^2 = \frac{\theta_\infty^2 (x - q_0)^2 P_2(x)}{4(x - t)^2 x^2 (x - 1)^2} \text{ with } P_2(x) = x^2 + \left(-\frac{\tilde{\theta}_0^2 t(t + 1)}{q_0^2} + \frac{\tilde{\theta}_1^2 t(t - 1)}{(q_0 - 1)^2} - \frac{\tilde{\theta}_t^2 t(t - 1)}{(q_0 - t)^2} \right) x + \frac{\tilde{\theta}_0^2 t^2}{q_0^2}. \quad (\text{D.12})$$

Equivalently it is defined by the following 3 conditions:

$$P_2(0) = \frac{\tilde{\theta}_0^2 t^2}{q_0^2}, \quad P_2(1) = \frac{(t - 1)^2 \tilde{\theta}_1^2}{(q_0 - 1)^2} \text{ and } P_2(t) = \frac{t^2 (t - 1)^2 \tilde{\theta}_t^2}{(q_0 - t)^2}. \quad (\text{D.13})$$

so that:

$$P_2(x) = \frac{\tilde{\theta}_0^2 t}{q_0^2} (x - 1)(x - t) - \frac{(t - 1) \tilde{\theta}_1^2}{(q_0 - 1)^2} x(x - t) + \frac{t(t - 1) \tilde{\theta}_t^2}{(q_0 - t)^2} x(x - 1). \quad (\text{D.14})$$

We now discuss the general form of the spectral curve and its possible degeneracies.

- The spectral curve is generically of genus 0 since the numerator admits a double zero at $x = q_0$.
- From (D.13) we observe that $P_2(x)$ cannot have zeros at $x \in \{0, 1, t\}$ when the monodromy parameters θ_0, θ_1 and θ_t are non-vanishing. Additionally, $P_2(x)$ cannot have a double zero since in that case, it would lead to the fact that $1 \pm \frac{\tilde{\theta}_0 t}{q_0} \pm \frac{\tilde{\theta}_1 (t - 1)}{q_0 - 1} = 0$ in contradiction with (C.7) when the monodromies are non-vanishing.
- The simple poles of $P_2(x)$ can never coincide. Indeed, in that case we would get from (D.11) that:

$$1 + \epsilon_0 \frac{\theta_0 t}{\theta_\infty q_0} + \epsilon_1 \frac{\theta_1 (t - 1)}{\theta_\infty (q_0 - 1)} = 0 \text{ with } \epsilon_0, \epsilon_1 \in \{-1, +1\}$$

Extracting t from this equation and inserting it back into the algebraic equation satisfied by q_0 (C.7) leads to:

$$\theta_\infty^2 (\epsilon_0 \theta_0 + \epsilon_1 \theta_1) \left((\epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \theta_\infty)^2 - \theta_t^2 \right) \left(q_0 + \frac{\epsilon_0 \theta_0}{\theta_\infty} \right) \left(q_0 - 1 - \frac{\epsilon_1 \theta_1}{\theta_\infty} \right) \left(q_0 - \frac{\epsilon_0 \theta_0}{\epsilon_0 \theta_0 + \epsilon_1 \theta_1} \right) = 0$$

Note also that $q_0 = \frac{\epsilon_0 \theta_0}{\epsilon_0 \theta_0 + \epsilon_1 \theta_1}$ is equivalent to $t = \infty$ or $\theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 = 0$. Similarly, $q_0 = -\frac{\epsilon_0 \theta_0}{\theta_\infty}$ is equivalent to $t = 1$ and $q_0 = 1 + \frac{\epsilon_1 \theta_1}{\theta_\infty}$ is equivalent to $t = 1$ both of which have been ruled out before. Consequently as soon as:

$$\theta_0^2 \neq \theta_1^2, \theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 \neq 0 \text{ and } \theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \epsilon_t \theta_t \neq 0 \quad (\text{D.15})$$

for any choice of the signs $(\epsilon_0, \epsilon_1, \epsilon_t) \in \{-1, +1\}^3$ then we can rule out that the simple zeros coincide at any time. This corresponds to the assumptions made for non-singular monodromies.

- The zeros of $P_2(x)$ can never coincide with q_0 . Indeed from (D.14), that would imply:

$$\frac{\tilde{\theta}_0^2 t}{q_0^3} - \frac{(t-1)\tilde{\theta}_1^2}{(q_0-1)^3} + \frac{t(t-1)\tilde{\theta}_t^2}{(q_0-t)^3} = 0$$

which precisely corresponds to a singular time (3.13).

E Proof of the existence of a formal series expansion for W_n

In this section we prove that the first condition required for the Topological Type property holds in the six Painlevé cases defined from (2.1)~(2.6). We have assumed in assumption 3.1 that the solutions $q_J(t)$ of the Painlevé equations admit a formal series expansion in \hbar . Since $M(x, t)$ satisfies:

$$\begin{aligned} \hbar \partial_x M(x, t) &= [\mathcal{D}(x, t), M(x, t)] \\ \hbar \partial_t M(x, t) &= [\mathcal{R}(x, t), M(x, t)] \end{aligned} \quad (\text{E.1})$$

and because we know that $\mathcal{D}(x, t)$ and $\mathcal{R}(x, t)$ also have a series expansion in \hbar from (3.2) we get that $M(x, t)$ admits a formal series expansion in \hbar of the form:

$$M(x, t) = \begin{pmatrix} \sum_{k=0}^{\infty} M_{1,1}^{(k)}(x, t) \hbar^k & \sum_{k=0}^{\infty} M_{1,2}^{(k)}(x, t) \hbar^k \\ \sum_{k=0}^{\infty} M_{2,1}^{(k)}(x, t) \hbar^k & 1 - \sum_{k=0}^{\infty} M_{1,1}^{(k)}(x, t) \hbar^k \end{pmatrix} \quad (\text{E.2})$$

From the alternative definition (6.7) of the correlation functions W_n we get that the correlation functions also have a series expansion in \hbar :

$$\forall n \geq 2 : W_n(x_1, \dots, x_n) = \sum_{k=0}^{\infty} W_n^{(k)}(x_1, \dots, x_n) \hbar^k \quad (\text{E.3})$$

Note that if we perform an admissible gauge transformation of type $\tilde{\Psi}(x, t) = U(t, \hbar) \Psi(x, t)$ then $\mathcal{D}(x, t)$, $\mathcal{R}(x, t)$ and $M(x, t)$ may have a much more complicated \hbar series expansion if $U(t, \hbar)$ depends on \hbar in a non-trivial way. In particular $M(x, t)$ may not have a series expansion at all. However, since the correlation functions are defined as traces of products of such matrices, then (E.3) still holds in any admissible gauge.

F Proof of the parity property

We want to prove in this section that the \hbar series expansion of the correlation functions $W_n(x_1, \dots, x_n)$ only involves powers of \hbar of the same parity as n (i.e. we want to explain why we have an exponent $2g$ and not only g in (6.13)). In order to do this, we use Proposition 3.3 of [2] that provides a sufficient criteria to obtain the $\hbar \leftrightarrow -\hbar$ symmetry. We recall their proposition here:

Proposition F.1 (Proposition 3.3 of [2]) *Let us denote \dagger the operator switching \hbar into $-\hbar$. If there exists an invertible matrix $\Gamma(t)$ independent of x such that:*

$$\Gamma^{-1}(t)\mathcal{D}^t(x, t)\Gamma(t) = \mathcal{D}^\dagger(x, t) \quad (\text{F.1})$$

then the correlation functions W_n satisfy:

$$\forall n \geq 1 : W_n^\dagger = (-1)^n W_n \quad (\text{F.2})$$

In particular if this proposition is satisfied then it automatically follows that the \hbar expansion of a given function $W_n(x_1, \dots, x_n)$ may only involve powers of \hbar with the same parity (given by the parity of n). Therefore all we have to do is prove the existence of a suitable matrix $\Gamma(t)$ in our six Painlevé cases. Since we know from (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) the expression of (p^\dagger, q^\dagger) in terms of (p, q) it is straightforward to compute the various $\Gamma(t)$ matrices.

Theorem F.2 (Parity Property) *In all six Painlevé cases, there exists a matrix $\Gamma_J(t)$ ($1 \leq J \leq 6$) such that*

$$\Gamma_J^{-1}(t)\mathcal{D}_J^t(x, t)\Gamma_J(t) = \mathcal{D}_J^\dagger(x, t)$$

The corresponding matrices are the following:

- Painlevé I: $\Gamma_{\text{I}}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Painlevé II: $\Gamma_{\text{II}}(t) = \begin{pmatrix} -2p & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé III: $\Gamma_{\text{III}}(t) = \begin{pmatrix} -\frac{p-t}{t} & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé IV: $\Gamma_{\text{IV}}(t) = \begin{pmatrix} -2(pq + \theta_0 + \theta_\infty) & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé V: $\Gamma_{\text{V}}(t) = \begin{pmatrix} -\frac{pq}{pq+\theta_0} & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé VI: $\Gamma_{\text{VI}}(t) = \begin{pmatrix} -\frac{t^2 z_0(z_0+\theta_0)}{q} + \frac{(t-1)^2 z_1(z_1+\theta_1)}{q-1} & 0 \\ 0 & 1 \end{pmatrix}$

We remark that the determination of $\Gamma(t)$ is not unique. Indeed, it is determined up to a global constant since multiplying $\Gamma(t)$ by a constant does not change the product $\Gamma^{-1}(t)\mathcal{D}^t(x,t)\Gamma(t)$. To fix this degree of freedom, we chose to fix one entry to 1 (usually the entry (2, 2) except for Painlevé I). We also observe that in all six cases: $\Gamma^t(t) = \Gamma(t)$ and $\Gamma^\dagger(t) = \Gamma(t)$. Note that the parity property is obviously invariant under admissible gauge transformations because the correlation functions are invariant (except W_1 that may be shifted (6.10) but for which the result holds too). However since the existence of the $\Gamma(t)$ matrix is only a sufficient condition to prove the parity property, it is a natural question to wonder if the existence of a Γ matrix satisfying (F.2) is independent of the choice of an admissible gauge. The answer to this question is positive. In fact gauge transformations of type (2.29) lead to $\tilde{\Gamma}(t) = (U^{-1})^t \Gamma(t) (U^\dagger)^{-1}$ while gauge transformations of type (2.28) lead to $\tilde{\Gamma}(t) = \Gamma(t)$.

G Proof of the pole structure

We want to prove in this section that the correlation functions defined through the determinantal formulas only have poles at the branchpoints of the spectral curve. In particular, we show that there is no singularities at the even zeros of the spectral curve or at the poles of the matrices $\mathcal{R}(x, t)$ and $\mathcal{D}(x, t)$ (i.e. depending on the case $x = 0$, $x = 1$ and/or $x = t$). First we note that this condition is not trivial since in all six Painlevé cases, the spectral curve admits a double zero. The idea of the proof follows the same spirit as the one proposed in appendix B of [19]. It consists in two steps:

1. Compute the matrix $M^{(0)}(x, t)$ and observe that it is regular at poles of $\mathcal{R}(x, t)$ and $\mathcal{D}(x, t)$ or at the even zeros of the spectral curve.
2. Compute the inverse of the matrix $\mathcal{R}^{(0)}(x, t)$ with which one can establish a recursive relation between $M^{(k)}(x, t)$ and lower orders $M^{(j)}(x, t)$ with $0 \leq j \leq k-1$ and their time derivatives.

Differential equations on $\Psi(x, t)$ defining the Lax pairs turns into the following system for $M(x, t)$:

$$\hbar \partial_x M(x, t) = [\mathcal{D}(x, t), M(x, t)] \quad \text{and} \quad \hbar \partial_t M(x, t) = [\mathcal{R}(x, t), M(x, t)] \quad (\text{G.1})$$

These equations will give a way to compute all orders $M^{(k)}(x, t)$.

G.1 Computation of $M^{(0)}(x, t)$

In full generality, the matrix $M^{(0)}(x, t)$ is characterized by the following set of equations:

$$[\mathcal{D}^{(0)}(x, t), M^{(0)}(x, t)] = 0 \quad \text{or} \quad [\mathcal{R}^{(0)}(x, t), M^{(0)}(x, t)] = 0 \quad (\text{G.2})$$

as well as $\text{Tr} M^{(0)}(x, t) = 1$ and $\det M^{(0)}(x, t) = 0$. Note that using the matrix $\mathcal{D}^{(0)}(x, t)$ or $\mathcal{R}^{(0)}(x, t)$ in the last equation is completely equivalent since the differential systems of the Lax pair are compatible (so that the system of equations is overdetermined but remain compatible). Thus if we denote $M^{(0)}(x, t) = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & 1 - m_{1,1} \end{pmatrix}$ a possible minimal set of equations is given by:

$$\begin{aligned} 0 &= \mathcal{R}_{2,1}^{(0)} m_{1,2} - \mathcal{R}_{1,2}^{(0)} m_{2,1} \\ 0 &= (2m_{1,1} - 1) \mathcal{R}_{1,2}^{(0)} - 2\mathcal{R}_{1,1}^{(0)} m_{1,2} \end{aligned}$$

$$0 = m_{1,1}(1 - m_{1,1}) - m_{1,2}m_{2,1} \quad (\text{G.3})$$

This system can be solved explicitly by:

$$M^{(0)}(x, t) = \begin{pmatrix} \frac{1}{2} + \frac{\mathcal{R}_{1,1}^{(0)}(x, t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x, t)}} & \frac{\mathcal{R}_{1,2}^{(0)}(x, t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x, t)}} \\ \frac{\mathcal{R}_{2,1}^{(0)}(x, t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x, t)}} & \frac{1}{2} - \frac{\mathcal{R}_{1,1}^{(0)}(x, t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x, t)}} \end{pmatrix} \quad (\text{G.4})$$

Note that one can replace the matrix $\mathcal{R}^{(0)}$ by $\mathcal{D}^{(0)}$ without changing the solution. From the definition of the Lax pairs, the entries $\mathcal{R}_{i,j}^{(0)}(x, t)$ may be singular, but that is only the case for Painlevé III and VI. In those cases, the poles of $\mathcal{R}_{i,j}^{(0)}(x, t)$ cancel out with the determinants of $\mathcal{R}^{(0)}(x, t)$ that we give below. Consequently, the matrix $M^{(0)}(x, t)$ is singular only at the points where the determinant $\mathcal{R}^{(0)}(x, t)$ vanishes. It is long but straightforward computations to get these determinants in all six Painlevé cases:

$$\begin{aligned} (P_I) &: \det \mathcal{R}_I^{(0)} = -(x + 2q_0) \\ (P_{II}) &: \det \mathcal{R}_{II}^{(0)} = -\frac{1}{4} \left(x^2 + 2q_0x + q_0^2 + \frac{\theta}{q_0} \right) \\ (P_{III}) &: \det \mathcal{R}_{III}^{(0)} = -\frac{(q_0x-1)^2((\theta_\infty - \theta_0q_0^2)x^2 - 2xq_0(\theta_\infty q_0^2 - \theta_0) + q_0^2(\theta_\infty - \theta_0q_0^2))}{4x^2q_0^2(\theta_\infty - \theta_0q_0^2)} \\ (P_{IV}) &: \det \mathcal{R}_{IV}^{(0)} = q_0^2 \left(x^2 + 2(q_0 + t)x + \frac{\theta_0^2}{q_0^2} \right) \\ (P_V) &: \det \mathcal{R}_V^{(0)} = -\frac{1}{4}(x - Q_1)(x - Q_2) \\ &\quad \text{where } Q_1 \text{ and } Q_2 \text{ are the simple zeros of the spectral curve.} \\ (P_{VI}) &: \det \mathcal{R}_{VI}^{(0)} = -\frac{(q_0-t)^2\theta_\infty^2 P_2(x)}{4t^2(t-1)^2(x-t)^2} \\ &\quad \text{where } P_2(x) \text{ is the monic polynomial of degree 2 appearing in the spectral curve.} \end{aligned}$$

Table 2: List of $\det \mathcal{R}_j^{(0)}(x, t)$

One can observe that these determinants only involve the simple zeros (i.e. the branchpoints) of the spectral curve but never vanish at the double zero of the spectral curve. Consequently we get that in all six Painlevé cases, the matrix $M^{(0)}(x, t)$ is only singular at the branchpoints of the spectral curve. **From expression (G.4), direct computations from the definition (6.7) of $W_2^{(0)}(x_1, x_2)$ show that in all six cases we have $W_2^{(0)}(x(z_1), x(z_2))dx(z_1)dx(z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$.**

Remark G.1 One could replace $\mathcal{R}^{(0)}(x, t)$ by $\mathcal{D}^{(0)}(x, t)$ everywhere in (G.4) and still obtain the same expression for $M^{(0)}(x, t)$. However by doing so, we see that the discussion about a possible singularity at a double zero of the spectral curve is not obvious because the denominator is vanishing there. One would have to prove that for any of the four entries, the numerator also vanishes at the double zero, which is far from obvious.

G.2 Recursive system for higher orders

Let us first start with the observation that the entries $x \mapsto \mathcal{R}_{i,j}^{(k)}(x, t)$ for $k \geq 1$ are trivially regular in all Painlevé cases except for Painlevé III and VI. Indeed, these entries are only singular at $x = 0$ for Painlevé 3 and at $x = t$ for Painlevé 6. Let us now look at order \hbar^k with $k \geq 1$ of (G.1) and in $\det M(x, t) = 0$. We get:

$$[\mathcal{R}^{(0)}(x, t), M^{(k)}(x, t)] = \partial_t M^{(k-1)}(x, t) - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]$$

$$\begin{aligned}
& M^{(k-1)}(x, t)_{1,1} (1 - 2M^{(k-1)}(x, t)_{1,1}) - M^{(0)}(x, t)_{2,1} M^{(k)}(x, t)_{1,2} - M^{(0)}(x, t)_{1,2} M^{(k)}(x, t)_{2,1} \\
&= \sum_{i=1}^{k-1} (M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1}). \tag{G.5}
\end{aligned}$$

The first matrix equation provides two independent scalar equations and thus we get a 3×3 linear system that can be written in the following matrix form:

$$\begin{pmatrix} 0 & -\mathcal{R}_{2,1}^{(0)} & \mathcal{R}_{1,2}^{(0)} \\ -2\mathcal{R}_{1,2}^{(0)} & 2\mathcal{R}_{1,1}^{(0)} & 0 \\ 2M_{1,1}^{(0)} - 1 & M_{2,1}^{(0)} & M_{1,2}^{(0)} \end{pmatrix} \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} \partial_t M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \\ \partial_t M^{(k-1)}(x, t)_{1,2} - \mathcal{R}_{1,2}^{(k)} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \\ \sum_{i=1}^{k-1} (M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1}) \end{pmatrix} \tag{G.6}$$

Using the exact expression for $M^{(0)}(x, t)$ we get:

$$\begin{pmatrix} 0 & -\mathcal{R}_{2,1}^{(0)} & \mathcal{R}_{1,2}^{(0)} \\ -2\mathcal{R}_{1,2}^{(0)} & 2\mathcal{R}_{1,1}^{(0)} & 0 \\ \mathcal{R}_{1,1}^{(0)} & \frac{1}{2}\mathcal{R}_{2,1}^{(0)} & \frac{1}{2}\mathcal{R}_{1,2}^{(0)} \end{pmatrix} \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} \partial_t M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \\ \partial_t M^{(k-1)}(x, t)_{1,2} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \\ \sqrt{-\det \mathcal{R}^{(0)}} \sum_{i=1}^{k-1} (M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1}) \end{pmatrix} \tag{G.7}$$

Note in particular that the 3×3 matrix on the l.h.s. does not depend on the order k we consider (it is only \hbar^0 terms). In general inverting a matrix may create poles at the zeros of the determinant of the matrix (this is obvious if one uses the definition of the inverse using the matrix of cofactors). However in our case we have:

$$\det \begin{pmatrix} 0 & -\mathcal{R}_{2,1}^{(0)} & \mathcal{R}_{1,2}^{(0)} \\ -2\mathcal{R}_{1,2}^{(0)} & 2\mathcal{R}_{1,1}^{(0)} & 0 \\ \mathcal{R}_{1,1}^{(0)} & \frac{1}{2}\mathcal{R}_{2,1}^{(0)} & \frac{1}{2}\mathcal{R}_{1,2}^{(0)} \end{pmatrix} = -2\mathcal{R}_{1,2}^{(0)}(x, t) \det \mathcal{R}^{(0)}(x, t) \tag{G.8}$$

Consequently the inverse of the matrix will only have singularities at the zeros of the former determinant and at the singularities of the entries of $\mathcal{R}^{(0)}$. We have also seen earlier that the entries of $\mathcal{R}^{(0)}$ are regular, except for Painlevé III and VI. In exceptional cases, one has that the zeros of $[-2\mathcal{R}_{1,2}^{(0)}(x, t) \det \mathcal{R}^{(0)}(x, t)]^{-1}$ cancel out the poles of the entries of $\mathcal{R}^{(0)}$. Also, as we noted before, $\det \mathcal{R}^{(0)}(x, t)$ only vanishes at the branchpoints of the spectral curve, as we wish. Thus, the only singularities that may arise now are where the term $\mathcal{R}_{1,2}^{(0)}(x, t)$ vanishes. Again, from the definition, we see that this term is actually independent of x and it doesn't vanish for any Painlevé cases, except for Painlevé I and III. For Painlevé I, $\mathcal{R}_{1,2}^{(0)}(x, t)$ vanishes at a branch point, as we wish. For Painlevé III, the zero from $\mathcal{R}_{1,2}^{(0)}(x, t)$ cancels out with a pole from $\det \mathcal{R}^{(0)}(x, t)$.

Thus, one can proceed to a simple recursion to prove that $x \mapsto M^{(k)}(x, t)$ only has poles at branchpoints of the spectral curve.

Theorem G.2 *The function $x \mapsto M^{(k)}(x, t)$ only has poles at the branchpoints of the spectral curve for any $k \geq 0$.*

From the last theorem and the alternative definition of the correlation functions (6.7), it is then obvious that the correlation functions $W_n^{(g)}$ only have poles at the branchpoints of the spectral curve for $(g, n) \neq (0, 1), (0, 2)$, and $W_2^{(0)}$ only has a double pole along the diagonal and no other poles.

Remark G.3 *The proof of the pole structure presented here is only valid for our very specific gauge choice in which $\mathcal{D}(x, t)$, $\mathcal{R}(x, t)$ and $M(x, t)$ have series expansion in \hbar . Indeed, as seen earlier, if we perform a gauge transformation of the form $\tilde{\Psi}(x, t) = U(t, \hbar)\Psi(x, t)$ where $U(t, \hbar)$ has a singular behavior in \hbar then none of the matrices $\mathcal{D}(x, t)$, $\mathcal{R}(x, t)$ or $M(x, t)$ will admit a series expansion in \hbar (making quantities like $M^{(0)}(x, t)$, $\mathcal{R}^{(0)}(x, t)$ ill-defined) and therefore the previous proof will fail. However, the final result, i.e. that the correlation functions $W_n^{(g)}$ only have poles at the branchpoints of the spectral curve remains valid since the spectral curve and the correlation functions are invariant under these gauge transformations. In other words, we chose a particular gauge in which the proof can be written more easily but the validity of the results holds for any admissible gauge.*

H Proof of the $O(\hbar^{n-2})$ -property

Let us start the proof with introducing some notations similar to [2] and [3]. We remind the reader that determinantal formulas (6.1) have been introduced so that they satisfy a set of equations known as loop equations. These loop equations (also known as Schwinger-Dyson equations) originate in random matrix theory where they are crucial. We recall here the main result of [3]:

Proposition H.1 (Theorem 2.9 of [3]) *Let us define the following functions (we denote by L_n the set of variables $\{x_1, \dots, x_n\}$):*

$$\begin{aligned}
P_1(x) &= \frac{1}{\hbar^2} \det \mathcal{D}(x, t) \\
P_2(x; x_2) &= \frac{1}{\hbar} \text{Tr} \left(\frac{\mathcal{D}(x, t) - \mathcal{D}(x_2, t) - (x - x_2)\mathcal{D}'(x_2, t)}{(x - x_2)^2} M(x_2) \right) \\
Q_{n+1}(x; L_n) &= \frac{1}{\hbar} \sum_{\sigma \in S_n} \frac{\text{Tr} (\mathcal{D}(x) M(x_{\sigma(1)}) \dots M(x_{\sigma(n)}))}{(x - x_{\sigma(1)})(x_{\sigma(1)} - x_{\sigma(2)}) \dots (x_{\sigma(n-1)} - x_{\sigma(n)})(x_{\sigma(n)} - x)} \\
P_{n+1}(x; L_n) &= (-1)^n \left[Q_{n+1}(x; L_n) - \sum_{j=1}^n \frac{1}{x - x_j} \text{Res}_{x' \rightarrow x_j} Q_{n+1}(x', L_n) \right] \tag{H.1}
\end{aligned}$$

Then the correlation functions satisfy

$$P_1(x) = W_2(x, x) + W_1(x)^2, \tag{H.2}$$

and for $n \geq 1$:

$$\begin{aligned}
0 &= P_{n+1}(x; L_n) + W_{n+2}(x, x, L_n) + 2W_1(x)W_{n+1}(x, L_n) + \\
&\quad \sum_{J \subset L_n, J \neq \{\emptyset, L_n\}} W_{1+|J|}(x, J) W_{1+n-|J|}(x, L_n \setminus J) \\
&\quad + \sum_{j=1}^n \frac{d}{dx_j} \frac{W_n(x, L_n \setminus x_j) - W_n(L_n)}{x - x_j} \tag{H.3}
\end{aligned}$$

Moreover $P_{n+1}(x; L_n)$ is a rational function of x whose poles are at the poles of $\mathcal{D}(x, t)$.

The equations (H.2) and (H.3) are called loop equations. As we will see this proposition and a subtle induction are sufficient to prove that W_n is at least of order \hbar^{n-2} as developed in [19]. Let us now analyze the different possible poles of $P_{n+1}(x, L_n)$ using proposition H.1. We get the important theorem:

Theorem H.2 (Pole Structure of $P_{n+1}(x, L_n)$) *For any of the six Painlevé cases we have:*

- For Painlevé I and Painlevé II, $x \mapsto P_{n+1}(x, L_n)$ does not depend on x .
- For Painlevé III, $x \mapsto P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x^2}$.
- For Painlevé IV, $x \mapsto P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x}$.
- For Painlevé V, $x \mapsto P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x(x-1)}$.
- For Painlevé VI, $x \mapsto P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x(x-1)(x-t)}$.

proof:

The proof of the previous theorem is based on the evaluation of the different orders of singularity of $x \mapsto P_{n+1}(x, L_n)$ at the finite possible singularities and at $x = \infty$ from definition (H.1).

- For Painlevé I and II, $x \mapsto \mathcal{D}(x, t)$ does not have finite singularities and therefore $x \mapsto P_{n+1}(x, L_n)$ is a polynomial of x . However from its definition (H.1) we see that $P_{n+1}(x, L_n) \underset{x \rightarrow \infty}{=} O(1)$ so that it cannot depend on x .
- For Painlevé III, $x \mapsto \mathcal{D}(x, t)$ has a double pole at $x = 0$ so that $P_{n+1}(x, L_n)$ is a rational function of x with only a possible double pole at $x = 0$ and a pole at $x = \infty$. Moreover from its definition (H.1), we see that $P_{n+1}(x, L_n) \underset{x \rightarrow \infty}{=} O\left(\frac{1}{x^2}\right)$. Therefore the only possible case is that $P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x^2}$.
- For Painlevé IV, $x \mapsto \mathcal{D}(x, t)$ has a simple pole singularity at $x = 0$ and from (H.1), the behavior of $P_{n+1}(x, L_n)$ is of the form $P_{n+1}(x, L_n) \underset{x \rightarrow \infty}{=} O\left(\frac{1}{x}\right)$. Therefore the only possible case is that $P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x}$.
- For Painlevé V, $x \mapsto \mathcal{D}(x, t)$ has a simple pole singularity at $x = 0$ and $x = 1$. From (H.1), the behavior at infinity of $P_{n+1}(x, L_n)$ is of the form $P_{n+1}(x, L_n) \underset{x \rightarrow \infty}{=} O\left(\frac{1}{x^2}\right)$. Consequently the only possible solution is that $P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x(x-1)}$.
- For Painlevé VI, $x \mapsto \mathcal{D}(x, t)$ has a simple pole singularity at $x = 0$, $x = 1$ and $x = t$. From (H.1), the behavior at infinity of $P_{n+1}(x, L_n)$ is of the form $P_{n+1}(x, L_n) \underset{x \rightarrow \infty}{=} O\left(\frac{1}{x^3}\right)$. Consequently the only possible solution is that $P_{n+1}(x, L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x(x-1)(x-t)}$.

□

As in [19], the last theorem is sufficient to prove by induction that the leading order of $W_n(x_1, \dots, x_n)$ is at least of order \hbar^{n-2} . As we will see, the induction is very similar in the six cases and the last theorem is used only at very specific places. Let us define the following statement:

$$\mathcal{P}_k : W_j(x_1, \dots, x_j) \text{ is at least of order } \hbar^{k-2} \quad \text{for } j \geq k. \quad (\text{H.4})$$

The statement is obviously true for $k = 1$ and $k = 2$ from the definitions. Let us assume that the statement \mathcal{P}_i is true for all $i \leq n$. Now we look at the loop equation (H.3). By induction assumption, we have that the last two terms are at least of order \hbar^{n-2} . Indeed in the sum we have terms of order $\hbar^{1+|J|-2+1+n-|J|-2} = \hbar^{n-2}$. Moreover we also have from the same assumption that

$W_{n+2}(x, x, L_n)$ is also of order at least \hbar^{n-2} (since $n+2 \geq n$). Therefore $W_{n+1}(x, L_n)$ is at least of order \hbar^{n-2} , and by considering the coefficients of the \hbar^{n-3} in (H.3) we have:

$$0 = P_{n+1}^{(n-3)}(x; L_n) + 2W_1^{(-1)}(x)W_{n+1}^{(n-2)}(x, L_n) \quad (\text{H.5})$$

If we assume that $W_{n+1}^{(n-2)}(x, L_n) \neq 0$ then we have:

$$W_{n+1}^{(n-2)}(x, L_n) = \frac{P_{n+1}^{(n-3)}(x; L_n)}{2W_1^{(-1)}(x)} \quad (\text{H.6})$$

Since by definition $W_1^{(-1)}(x)$ is the spectral curve of the system, we get in our six cases:

- For Painlevé I:

$$W_{n+1}^{(n-2)}(x, L_n) = \frac{P_{n+1}^{(n-3)}(L_n)}{4(x - q_0)\sqrt{x + 2q_0}}$$

- For Painlevé II:

$$W_{n+1}^{(n-2)}(x, L_n) = \frac{P_{n+1}^{(n-3)}(L_n)}{2(x - q_0)\sqrt{x^2 + 2q_0x + q_0^2 + \frac{\theta}{q_0}}}$$

- For Painlevé III:

$$W_{n+1}^{(n-2)}(x, L_n) = \frac{\sqrt{q_0(q_0^4 - 1)}P_{n+1}^{(n-3)}(L_n)}{\sqrt{t}(q_0x + 1)\sqrt{(\theta_\infty - \theta_0q_0^2)x^2 - 2xq_0(\theta_\infty q_0^2 - \theta_0) + q_0^2(\theta_\infty - \theta_0q_0^2)}}$$

- For Painlevé IV:

$$W_{n+1}^{(n-2)}(x, L_n) = \frac{P_{n+1}^{(n-3)}(L_n)}{2(x - q_0)\sqrt{x^2 + 2(q_0 + t)x + \frac{\theta_0^2}{q_0}}}$$

- For Painlevé V:

$$W_{n+1}^{(n-2)}(x, L_n) = \frac{P_{n+1}^{(n-3)}(L_n)}{t(x - Q_0)\sqrt{(x - Q_1)(x - Q_2)}}$$

- For Painlevé VI:

$$W_{n+1}^{(n-2)}(x, L_n) = \frac{P_{n+1}^{(n-3)}(L_n)}{\theta_\infty(x - q_0)\sqrt{x^2 + \left(-1 - \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2} + \frac{\theta_1^2 (t-1)^2}{\theta_\infty^2 (q_0-1)^2}\right)x + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}}}$$

Now we observe that in all cases, we get that $x \mapsto W_{n+1}^{(n-2)}(x, L_n)$ must have a simple pole at the double zero of the spectral curve (i.e. q_0 for Painlevé I, II, IV, VI and $-\frac{1}{q_0}$ for Painlevé III and Q_0 for Painlevé V). This is in contradiction with the pole structure of the correlation functions proved in appendix G. Consequently we must have $W_{n+1}^{(n-2)}(x, L_n) = 0$. This proves that $W_{n+1}(x, L_n)$ is at least of order \hbar^{n-1} . We now need to prove the same statement for higher correlation functions. Let us prove it by a second induction by defining:

$$\tilde{\mathcal{P}}_i : W_i(x_1, \dots, x_i) \text{ is of order at least } \hbar^{n-1}. \quad (\text{H.7})$$

We want to prove $\tilde{\mathcal{P}}_i$ for all $i \geq n+1$ by induction. We just proved it for $i = n+1$ so initialization is done. Let us assume that $\tilde{\mathcal{P}}_j$ is true for all j satisfying $n+1 \leq j \leq i_0$. We look at the loop equation:

$$\begin{aligned} 0 &= P_{i_0+1}(x; L_{i_0}) + W_{i_0+2}(x, x, L_{i_0}) + 2W_1(x)W_{i_0+1}(x, L_{i_0}) \\ &\quad + \sum_{J \subset L_{i_0}, J \neq \{\emptyset, L_{i_0}\}} W_{1+|J|}(x, J)W_{1+i_0-|J|}(x, L_{i_0} \setminus J) \\ &\quad + \sum_{j=1}^{i_0} \frac{d}{dx_j} \frac{W_{i_0}(x, L_{i_0} \setminus x_j) - W_{i_0}(L_{i_0})}{x - x_j}. \end{aligned} \quad (\text{H.8})$$

By assumption on $\tilde{\mathcal{P}}_{i_0}$, the last sum with the derivatives contains terms of order at least \hbar^{n-1} . In the sum involving the subsets of L_{i_0} it is straightforward to see that the terms are all of order at least \hbar^{n-1} . Indeed, as soon as one of the index is greater than $n+1$, the assumption $\tilde{\mathcal{P}}_i$ for $n+1 \leq i \leq i_0$ tells us that this term is already at order at least \hbar^{n-1} . Since the second factor of the product is at least of order \hbar^0 then it does not decrease the order. Now if both factors have indexes strictly lower than $n+1$, then the assumption of \mathcal{P}_j for all $j \leq n$ tell us that the order of the product is at least of $\hbar^{|J|+1-2+1+i_0-|J|-2} = \hbar^{i_0-2}$ which is greater than $n-1$ since $i_0 \geq n+1$. Additionally by induction on \mathcal{P}_n we know that $W_{i_0+1}(x, L_{i_0})$ is at least of order \hbar^{n-2} as well as $W_{i_0+2}(x, x, L_{i_0})$. Consequently looking at order \hbar^{n-3} in (H.8) gives:

$$0 = P_{i_0+1}^{(n-3)}(x; L_{i_0}) + 2W_1^{(-1)}(x)W_{i_0+1}^{(n-2)}(x, L_{i_0}) \quad (\text{H.9})$$

We can apply a similar reasoning as the one developed for (H.5). If we assume $W_{i_0+1}^{(n-2)}(x, L_{i_0}) \neq 0$, then we have:

$$W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{P_{i_0+1}^{(n-3)}(x; L_{i_0})}{2W_1^{(-1)}(x)}. \quad (\text{H.10})$$

In our six cases we get:

- For Painlevé I:

$$W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{P_{i_0+1}^{(n-3)}(L_{i_0})}{4(x - q_0)\sqrt{x + 2q_0}}$$

- For Painlevé II:

$$W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{P_{i_0+1}^{(n-3)}(L_{i_0})}{2(x - q_0)\sqrt{x^2 + 2q_0x + q_0^2 + \frac{\theta}{q_0}}}$$

- For Painlevé III:

$$W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{\sqrt{q_0(q_0^4 - 1)}P_{i_0+1}^{(n-3)}(L_{i_0})}{\sqrt{t}(q_0x + 1)\sqrt{(\theta_\infty - \theta_0q_0^2)x^2 - 2xq_0(\theta_\infty q_0^2 - \theta_0) + q_0^2(\theta_\infty - \theta_0q_0^2)}}$$

- For Painlevé IV:

$$W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{P_{i_0+1}^{(n-3)}(L_{i_0})}{2(x - q_0)\sqrt{x^2 + 2(q_0 + t)x + \frac{\theta_0^2}{q_0^2}}}$$

- For Painlevé V:

$$W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{P_{i_0+1}^{(n-3)}(L_{i_0})}{t(x - Q_0)\sqrt{(x - Q_1)(x - Q_2)}}$$

- For Painlevé VI:

$$W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{P_{i_0+1}^{(n-3)}(L_{i_0})}{\theta_\infty(x - q_0)\sqrt{x^2 + \left(-1 - \frac{\theta_\infty^2 t^2}{\theta_\infty^2 q_0^2} + \frac{\theta_1^2(t-1)^2}{\theta_\infty^2(q_0-1)^2}\right)x + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}}}$$

In all cases we obtain that $x \mapsto W_{i_0+1}^{(n-2)}(x, L_{i_0})$ must have a simple pole at the double zero of the spectral curve in contradiction with the pole structure of the correlation functions proved in appendix G. Consequently we must have $W_{i_0+1}^{(n-2)}(x, L_{i_0}) = 0$. In particular it means that $W_{i_0+1}(x, L_{i_0})$ (which by assumption of \mathcal{P}_n was already known to be at least of order \hbar^{n-2}) is at least of order \hbar^{n-1} thus making the induction on $\tilde{\mathcal{P}}_{i_0}$. Hence by induction we have proved that $\forall i \geq n+1$, $\tilde{\mathcal{P}}_i$ holds which exactly proves that \mathcal{P}_{n+1} is true. Eventually by induction we have just proved that \mathcal{P}_n holds for $n \geq 1$. In other words, we have proved the leading order condition of the topological type property in our six cases.

Remark H.3 *As for the pole structure, the proof presented here is only valid in the gauge we selected. Indeed, the proof uses the existence of a nice \hbar series expansion for $\mathcal{D}(x, t)$, $\mathcal{R}(x, t)$ and $M(x, t)$ that may not exist after performing an admissible gauge transformation $\tilde{\Psi}(x, t) = U(t, \hbar)\Psi(x, t)$ with $U(t, \hbar)$ presenting a complicated \hbar dependence. However the final result (i.e. that the leading order of W_n is \hbar^{n-2}) remains valid in any admissible gauge transformations. since the correlation functions W_n are invariant under any admissible gauge transformations.*

I Computation of the free energies $F^{(2g)}$

I.1 Computation of $F^{(0)}$

The computation of $F^{(0)}$ requires specific computations detailed in [16]. We find the following results:

- Painlevé I: $d\omega(z) = Y(z)dx(z)$ has one singularity at $z = \infty$ (pole of $x(z)$ of order 4 in the language of [16]). The temperature t_∞ is vanishing and we find:

$$F_I^{(0)} = \frac{48 q_0^5}{5} \tag{I.1}$$

Note that we have $\dot{q}_0 = -\frac{1}{12q_0}$ so that $-\frac{d}{dt}F_I^{(0)} = 4q_0^3$ in agreement with $\frac{d}{dt}\tau_I^{(0)} = 4q_0^3$.

- Painlevé II: $d\omega(z) = Y(z)dx(z)$ has two singularities at $z = 0$ and $z = \infty$. We get (see [19]):

$$F_{II}^{(0)} = \frac{4\theta q_0^3}{3} + \frac{\theta^3}{24q_0^3} + \frac{\theta^2}{2} \ln q_0 - \frac{\theta^2}{4} - \frac{\theta^2}{2} \ln \theta + \theta^2 \ln 2 \tag{I.2}$$

We verify that $-\frac{d}{dt}F_{II}^{(0)} = \frac{\theta(8q_0^3 - \theta)}{8q_0^2} = \frac{d}{dt}\tau_{II}^{(0)}$.

- Painlevé III: $d\omega(z) = Y(z)dx(z)$ has two additional singularities at simple zeros of $x(z)$. These singularities are poles of order 2. We find:

$$F_{\text{III}}^{(0)} = \frac{1}{4}\theta_0\theta_\infty \ln\left(\frac{q_0^2+1}{q_0^2-1}\right) + \frac{\theta_0^2}{8} \ln\left(\frac{q_0^2(q_0^2\theta_0 - \theta_\infty)^2}{q_0^4-1}\right) + \frac{\theta_\infty^2}{8} \ln\left(\frac{(q_0^2\theta_0 - \theta_\infty)^2}{q_0^2(q_0^4-1)}\right) + \frac{3\theta_0^2 - 3\theta_\infty^2 - 2\theta_0\theta_\infty q_0^2 + (5\theta_\infty^2 - \theta_0^2)q_0^4 - 2\theta_0\theta_\infty q_0^6}{(q_0^4-1)^2} \quad (\text{I.3})$$

We verify that $-\frac{d}{dt}F_{\text{III}}^{(0)} = \frac{(\theta_0^2-\theta_\infty^2)-4\theta_0\theta_\infty q_0^2+4(\theta_\infty^2+\theta_0^2)q_0^4-4\theta_0\theta_\infty q_0^6+(\theta_\infty^2-\theta_0^2)q_0^8}{4q_0(q_0^4-1)(q_0^2\theta_0-\theta_\infty)} = \frac{d}{dt}\tau_{\text{III}}^{(0)}$

- Painlevé IV: $d\omega(z) = Y(z)dx(z)$ has four singularities at $z = 0$ and $z = \infty$ (poles of $x(z)$ of order 2) as well as zeros of $x(z)$ (poles of $Y(z)$). We get:

$$F_{\text{IV}}^{(0)} = \frac{(3q_0^4 - (8\theta_0 + \theta_\infty)q_0^2 + 2\theta_0^2)\sqrt{q_0^4 + 2\theta_\infty q_0^2 + \theta_0^2}}{2q_0^2} - \frac{3q_0^4}{2} + 5\theta_0 q_0^2 + \frac{\theta_0^3}{q_0^2} + \frac{(\theta_0^2 + \theta_\infty^2)}{2} \ln\left(q_0^2 + \theta_\infty^2 + \sqrt{q_0^4 + 2\theta_\infty q_0^2 + \theta_0^2}\right) - 2\theta_0(\theta_0 - \theta_\infty) \ln q_0 - \theta_0\theta_\infty \ln\left(2\theta_0^2 + 2\theta_\infty q_0^2 + 2\theta_0\sqrt{q_0^4 + 2\theta_\infty q_0^2 + \theta_0^2}\right) \quad (\text{I.4})$$

We can verify that $-\frac{d}{dt}F_{\text{IV}}^{(0)} = \frac{2(\theta_0 - q_0^2)(q_0^2 - \theta_0 - \sqrt{q_0^4 + 2\theta_\infty q_0^2 + \theta_0^2})}{q_0} = \frac{d}{dt}\tau_{\text{IV}}^{(0)}$.

- Painlevé V: In this case, it is easier to express all quantities in terms of the double zero Q_0 of the spectral curve. $d\omega(z) = Y(z)dx(z)$ has six singularities at $z = 0$ and $z = \infty$ and at the two conjugate zeros of $x(z)$ and $x(z) - 1$. Tedious computations give:

$$F_{\text{V}}^{(0)} = (\theta_0^2 + \theta_1^2 + \theta_\infty^2) \left(\frac{1}{2} \ln 2 - \frac{1}{8} \ln \left(\prod_{\pm} \left(1 \pm \frac{\theta_0}{tQ_0} \pm \frac{\theta_1}{t(Q_0-1)} \right) \right) \right) + \frac{\theta_0^2}{2} \ln \theta_0 + \frac{\theta_1^2}{2} \ln \theta_1 - \frac{\theta_0^2}{2} \ln Q_0 - \frac{\theta_1^2}{2} \ln(Q_0 - 1) - \frac{\theta_0^2 + \theta_1^2}{2} \ln t + \frac{\theta_0\theta_\infty}{4} \ln \left(\frac{\left(1 - \frac{\theta_0}{tQ_0} + \frac{\theta_1}{t(Q_0-1)}\right) \left(1 - \frac{\theta_0}{tQ_0} - \frac{\theta_1}{t(Q_0-1)}\right)}{\left(1 + \frac{\theta_0}{tQ_0} + \frac{\theta_1}{t(Q_0-1)}\right) \left(1 + \frac{\theta_0}{tQ_0} - \frac{\theta_1}{t(Q_0-1)}\right)} \right) + \frac{\theta_1\theta_\infty}{4} \ln \left(\frac{\left(1 + \frac{\theta_0}{tQ_0} - \frac{\theta_1}{t(Q_0-1)}\right) \left(1 - \frac{\theta_0}{tQ_0} - \frac{\theta_1}{t(Q_0-1)}\right)}{\left(1 + \frac{\theta_0}{tQ_0} + \frac{\theta_1}{t(Q_0-1)}\right) \left(1 - \frac{\theta_0}{tQ_0} + \frac{\theta_1}{t(Q_0-1)}\right)} \right) + \frac{\theta_0\theta_1}{4} \ln \left(\frac{\left(1 - \frac{\theta_0}{tQ_0} + \frac{\theta_1}{t(Q_0-1)}\right) \left(1 + \frac{\theta_0}{tQ_0} - \frac{\theta_1}{t(Q_0-1)}\right)}{\left(1 + \frac{\theta_0}{tQ_0} + \frac{\theta_1}{t(Q_0-1)}\right) \left(1 - \frac{\theta_0}{tQ_0} - \frac{\theta_1}{t(Q_0-1)}\right)} \right) - \frac{\theta_0^2}{2Q_0} + \frac{\theta_1^2}{2(Q_0-1)} + \frac{1}{4}t\theta_\infty \left(1 + \frac{\theta_0^2}{t^2Q_0^2} - \frac{\theta_1^2}{t^2(Q_0-1)^2} \right) - \frac{t^2}{32} \prod_{\pm} \left(1 \pm \frac{\theta_0}{tQ_0} \pm \frac{\theta_1}{t(Q_0-1)} \right) \quad (\text{I.5})$$

where the product \prod_{\pm} is to be taken on the four possible choices of signs. Moreover we get from the Lax pair:

$$\frac{d}{dt}\tau_{\text{V}}^{(0)} = \frac{(\theta_0 + \theta_1 - \theta_\infty)q_0^2 + \theta_0 + \theta_1 + \theta_\infty}{4(q_0^2 - 1)} - \frac{((\theta_0 - \theta_1 - \theta_\infty)q_0^2 - \theta_0 + \theta_1 - \theta_\infty)((\theta_0 - \theta_1 - \theta_\infty)q_0 - \theta_0 + \theta_1 - \theta_\infty)}{8t(q_0 + 1)q_0} \quad (\text{I.6})$$

In order to compare it with the expression of $F_{\text{V}}^{(0)}$ we observe that we have:

$$q_0 = \frac{Q_0^2(\theta_0 - \theta_1) - \theta_0(2Q_0 - 1) + Q_0(Q_0 - 1)((2Q_0 - 1)t - \theta_\infty)}{Q_0(Q_0 - 1)(\theta_0 - \theta_1 - \theta_\infty)} \quad (\text{I.7})$$

and that following (C.5), (t, Q_0) satisfies the following algebraic equation:

$$Q_0^2(Q_0 - 1)^2(2Q_0 - 1)t^2 - 2\theta_\infty t Q_0^2(Q_0 - 1)^2 + (Q_0(\theta_0 + \theta_1) - \theta_0)(Q_0(\theta_0 - \theta_1) - \theta_0) = 0 \quad (\text{I.8})$$

Consequently we can express $\frac{d}{dt}\tau_V^{(0)}$ from (I.6) and (I.7) in terms of t and Q_0 and observe using (I.8) that $-\frac{d}{dt}F_V^{(0)} = \frac{d}{dt}\tau_V^{(0)}$.

- Painlevé VI: The long computation is detailed in section I.4 as illustrations of the method. We find:

$$\begin{aligned} F_{\text{VI}}^{(0)} = & \frac{\theta_0^2 + \theta_1^2 + \theta_t^2 + \theta_\infty^2}{2} \ln 2 - \frac{\theta_0^2 + \theta_1^2 + \theta_t^2}{2} \ln \theta_\infty + \frac{\theta_0^2}{2} \ln \theta_0 + \frac{\theta_1^2}{2} \ln \theta_1 + \frac{\theta_t^2}{2} \ln \theta_t \\ & - \frac{\theta_0^2}{2} \ln q_0 - \frac{\theta_1^2}{2} \ln(q_0 - 1) - \frac{\theta_t^2}{2} \ln(q_0 - t) + \frac{i\pi}{4}(\theta_0\theta_1 + \theta_0\theta_t + \theta_1\theta_t) \\ & + \left(\frac{\theta_0^2 + \theta_t^2}{2} - \frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{12} \right) \ln t + \left(\frac{\theta_1^2 + \theta_t^2}{2} - \frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{12} \right) \ln(t - 1) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} + \frac{\theta_0\theta_\infty}{8} + \frac{\theta_1\theta_\infty}{8} + \frac{\theta_0\theta_1}{4} \right) \ln \left(1 + \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} - \frac{\theta_0\theta_\infty}{8} - \frac{\theta_1\theta_\infty}{8} + \frac{\theta_0\theta_1}{4} \right) \ln \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} + \frac{\theta_0\theta_\infty}{8} - \frac{\theta_1\theta_\infty}{8} - \frac{\theta_0\theta_1}{4} \right) \ln \left(1 + \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} - \frac{\theta_0\theta_\infty}{8} + \frac{\theta_1\theta_\infty}{8} - \frac{\theta_0\theta_1}{4} \right) \ln \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} + \frac{\theta_0\theta_\infty}{8} - \frac{\theta_t\theta_\infty}{8} - \frac{\theta_0\theta_t}{4} \right) \ln \left(1 + \frac{\theta_0}{\theta_\infty q_0} + \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} - \frac{\theta_0\theta_\infty}{8} + \frac{\theta_t\theta_\infty}{8} - \frac{\theta_0\theta_t}{4} \right) \ln \left(1 - \frac{\theta_0}{\theta_\infty q_0} - \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} + \frac{\theta_0\theta_\infty}{8} + \frac{\theta_t\theta_\infty}{8} + \frac{\theta_0\theta_t}{4} \right) \ln \left(1 + \frac{\theta_0}{\theta_\infty q_0} - \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} - \frac{\theta_0\theta_\infty}{8} - \frac{\theta_t\theta_\infty}{8} + \frac{\theta_0\theta_t}{4} \right) \ln \left(1 - \frac{\theta_0}{\theta_\infty q_0} + \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} - \frac{\theta_1\theta_\infty}{8} - \frac{\theta_t\theta_\infty}{8} + \frac{\theta_1\theta_t}{4} \right) \ln \left(1 + \frac{\theta_1}{\theta_\infty(q_0-1)} + \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} + \frac{\theta_1\theta_\infty}{8} + \frac{\theta_t\theta_\infty}{8} + \frac{\theta_1\theta_t}{4} \right) \ln \left(1 - \frac{\theta_1}{\theta_\infty(q_0-1)} - \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} - \frac{\theta_1\theta_\infty}{8} + \frac{\theta_t\theta_\infty}{8} - \frac{\theta_1\theta_t}{4} \right) \ln \left(1 + \frac{\theta_1}{\theta_\infty(q_0-1)} - \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \\ & - \left(\frac{\theta_0^2 + \theta_1^2 + \theta_\infty^2 + \theta_t^2}{24} + \frac{\theta_1\theta_\infty}{8} - \frac{\theta_t\theta_\infty}{8} - \frac{\theta_1\theta_t}{4} \right) \ln \left(1 - \frac{\theta_1}{\theta_\infty(q_0-1)} + \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \end{aligned} \quad (\text{I.9})$$

We can verify that $-\frac{d}{dt}F_{\text{VI}}^{(0)} = \frac{d}{dt}\tau_{\text{VI}}^{(0)}$ with:

$$\frac{d}{dt}\tau_{\text{VI}}^{(0)} = \frac{(\theta_0^2 - (\theta_0^2 - \theta_1^2 + \theta_\infty^2 - \theta_t^2)q_0 + (\theta_\infty^2 - \theta_t^2)q_0^2)t(t - 2q_0) + q_0^2(\theta_0^2 + \theta_t^2 - (\theta_1^2 - \theta_0^2 - \theta_\infty^2 - \theta_t^2)q_0 + \theta_\infty^2 q_0^2)}{4t(t-1)q_0(q_0-1)(q_0-t)} \quad (\text{I.10})$$

I.2 Computation of $F^{(2)}$

One branchpoint case:

In the case of a parametrization of the form (5.5) $x(z) = z^2 + a$ and $Y(z) = zg(x(z))$ (i.e. $Y^2(x) = (x - a)g(x)^2$ with g a rational function that does not vanish at $x = a$) with only one branchpoint at $x = a$ (i.e. $z = 0$), the formula proposed by Eynard and Orantin (mind the different sign convention and the change $g \rightarrow 2g$ in our notation) for $F^{(2)}$ in [16] reduces to:

$$F_I^{(2)} = \frac{1}{24} \ln(y'(0)) = \frac{1}{24} \ln g(a) \quad (\text{I.11})$$

For Painlevé 1 we find $(g(x) = 2(x - q_0)$ and $a = -2q_0$):

$$F_I^{(2)} = \frac{1}{24} \ln(-6q_0) \text{ and } \frac{d}{dt} \tau_I^{(2)} = -\frac{d}{dt} F_I^{(2)} = -\frac{1}{288q_0^2} = \frac{1}{48t} \quad (\text{I.12})$$

We have used here the fact that $q_0^2 = -\frac{t}{6}$ to simplify quantities. Thus we have verified that $\frac{d}{dt} \tau_I^{(2)} = -\frac{d}{dt} F_I^{(2)}$

Two branchpoints case:

In the case of a parametrization of the form (5.6) $x(z) = \frac{a+b}{2} + \frac{b-a}{4} \left(z + \frac{1}{z}\right)$ and $Y(x) = \frac{(b-a)(z-1)(z+1)}{4z} g(x(z))$ (i.e. $Y^2(x) = (x - a)(x - b)g(x)^2$) with $g(x)$ a rational function in x not vanishing at $x = a$ and $x = b$, the formula proposed by Eynard and Orantin for $F^{(2)}$ in [16] reduces to:

$$F^{(2)} = \frac{1}{24} \ln(-(b - a)^4 g(a)g(b)) \quad (\text{note } \tau_{\text{Berg}} = (b - a)^{\frac{1}{4}}) \quad (\text{I.13})$$

It is also in agreement with the formula presented in [8]. Then it is straightforward to compute the values of $F^{(2)}$ in all six cases by inserting the values of a , b and $g(x)$ in the previous formula. We find:

- Painlevé II:

$$F_{\text{II}}^{(2)} = \frac{1}{24} \ln \left(-16\theta^2 \left(4 + \frac{\theta}{q_0^3} \right) \right) \text{ and } \frac{d}{dt} \tau_{\text{II}}^{(2)} = -\frac{d}{dt} F_{\text{II}}^{(2)} = -\frac{\theta q_0}{8(4q_0^3 + \theta)^2} \quad (\text{I.14})$$

where we used $t = -2q_0^2 + \frac{\theta}{q_0}$ to remove t from all previous quantities.

- Painlevé III:

$$\begin{aligned} F_{\text{III}}^{(2)} &= \frac{1}{24} \ln \left(\frac{4(\theta_\infty^2 - \theta_0^2)^2(\theta_0 q_0^6 - 3\theta_0 q_0^4 + 3\theta_0 q_0^2 - \theta_\infty)}{(\theta_\infty - \theta_0 q_0^2)^3} \right) \\ \frac{d}{dt} \tau_{\text{III}}^{(2)} &= -\frac{d}{dt} F_{\text{III}}^{(2)} = -\frac{(\theta_\infty^2 - \theta_0^2) q_0^3 (q_0^4 - 1)^2}{2(\theta_\infty - \theta_0 q_0^2)(\theta_0 q_0^6 - 3\theta_0 q_0^4 + 3\theta_0 q_0^2 - \theta_\infty)^2} \end{aligned} \quad (\text{I.15})$$

where we used $t = \frac{q_0(\theta_\infty - \theta_0 q_0^2)}{q_0^4 - 1}$ to remove t from all previous quantities.

- Painlevé IV:

$$\begin{aligned} F_{\text{IV}}^{(2)} &= \frac{1}{24} \ln \left(-\frac{16(\theta_0 - q_0^2 - tq_0)^2(\theta_0 + q_0^2 + tq_0)^2(3q_0^4 + 2q_0^3t + \theta_0^2)}{\theta_0^2 q_0^4} \right) \\ \frac{d}{dt} \tau_{\text{IV}}^{(2)} &= -\frac{d}{dt} F_{\text{IV}}^{(2)} = \frac{(\theta_0 - q_0^2 - tq_0)(\theta_0 + q_0^2 + tq_0)q_0^3}{4(3q_0^4 + 2q_0^3t + \theta_0^2)^2} \end{aligned} \quad (\text{I.16})$$

where we used $\theta_\infty = \frac{3q_0^4 + 4q_0^3t + tq_0^2 - \theta_0^2}{2q_0^2}$ to remove θ_∞ from all previous quantities.

- Painlevé V:

$$F_V^{(2)} = \frac{1}{24} \ln \left[-\frac{4((\theta_0 - \theta_1 - \theta_\infty)q_0^2 + 2(t + \theta_1 - \theta_0)q_0 + \theta_0 - \theta_1 + \theta_\infty)^2 P_4(t)}{(q_0 - 1)^2 ((q_0 - 1)^4 (\theta_0 + \theta_\infty - \theta_1 - (\theta_0 - \theta_1 - \theta_\infty)q_0)^2 - 4q_0^2(q_0 + 1)^2 t^2)} \right] \quad (\text{I.17})$$

where $P_4(t) = a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$ is given by:

$$\begin{aligned} a_4 &= 16q_0^4(6q_0 + q_0^2 + 1) \\ a_3 &= 32(q_0 - 1)q_0^3((\theta_0 + \theta_\infty - \theta_1) + (3\theta_\infty + \theta_0 - 5\theta_1)q_0 + (3\theta_\infty + \theta_1 - 5\theta_0)q_0^2 + (-\theta_0 + \theta_1 + \theta_\infty)q_0^3) \\ a_2 &= 8q_0^2(q_0 - 1)^2(3(-\theta_0 - \theta_\infty + \theta_1)^2 + 4(-\theta_0 - \theta_\infty + \theta_1)(2\theta_0 - \theta_\infty + 2\theta_1)q_0 + (2\theta_0^2 + 2\theta_1^2 + 28\theta_0\theta_1 + 10\theta_\infty^2)q_0^2 \\ &\quad + (-4(2\theta_0 - \theta_\infty + 2\theta_1)(-\theta_0 + \theta_1 + \theta_\infty))q_0^3 + 3(-\theta_0 + \theta_1 + \theta_\infty)^2 q_0^4) \\ a_1 &= 8q_0(q_0 - 1)^3((\theta_0 + \theta_\infty - \theta_1) + (-\theta_0 + \theta_1 + \theta_\infty)q_0)^2 \\ &\quad ((\theta_0 + \theta_\infty - \theta_1) + (-3\theta_0 - \theta_\infty - \theta_1)q_0 + (-\theta_\infty - \theta_0 - 3\theta_1)q_0^2 + (-\theta_0 + \theta_1 + \theta_\infty)q_0^3) \\ a_0 &= (q_0 - 1)^6((\theta_0 + \theta_\infty - \theta_1) + (-\theta_0 + \theta_1 + \theta_\infty)q_0) \end{aligned} \quad (\text{I.18})$$

One can verify with tedious computations and (C.5) that $\frac{d}{dt}\tau_V^{(2)} = -\frac{d}{dt}F_V^{(2)}$ holds.

- Painlevé VI:

$$\begin{aligned} F_{VI}^{(2)} &= -\frac{1}{12}(\ln 2 + \ln \theta_0 + \ln \theta_1 + \ln \theta_\infty + \ln \theta_t) \\ &\quad -\frac{1}{9}\ln t - \frac{1}{9}\ln(t-1) + \frac{1}{12}\ln q_0 + \frac{1}{12}\ln(q_0-1) + \frac{1}{12}\ln(q_0-t) \\ &\quad + \frac{1}{24}\ln\left(\theta_\infty^2 q_0^2 - q_0\left(\frac{\theta_0^2 t(t+1)}{q_0^2} - \frac{\theta_1^2 t(t-1)}{(q_0-1)^2} + \frac{\theta_t^2 t(t-1)}{(q_0-t)^2}\right) + \frac{\theta_0^2 t^2}{q_0^2}\right) \\ &\quad + \frac{1}{36}\ln\left(\prod_{\pm}\left(\theta_\infty \pm \frac{\theta_0 t}{q_0} \pm \frac{\theta_1(t-1)}{q_0-1}\right)\left(\theta_\infty \pm \frac{\theta_0}{q_0} \pm \frac{\theta_t(t-1)}{q_0-t}\right)\left(\theta_\infty \pm \frac{\theta_1}{q_0-1} \pm \frac{\theta_t t}{q_0-t}\right)\right) \end{aligned} \quad (\text{I.19})$$

where the product indexed by \pm indicates that we must take all possible choice of signs. One can verify with tedious computations and (C.7) that $\frac{d}{dt}\tau_{VI}^{(2)} = -\frac{d}{dt}F_{VI}^{(2)}$ holds.

I.3 Higher orders

For the simplest Painlevé equations, we can compute the first orders $F^{(4)}$, $F^{(6)}$, etc. of the topological recursion depending on the complexity of the spectral curve. We find:

- Painlevé I:

$$F_I^{(4)} = -\frac{7}{207360 q_0^5}, \quad F_I^{(6)} = -\frac{245}{429981696 q_0^{10}} \quad (\text{I.20})$$

in agreement ($\dot{q}_0 = -\frac{1}{12q_0}$) with $\frac{d}{dt}\tau_I^{(4)} = \frac{7}{497664 q_0^7}$ and $\frac{d}{dt}\tau_I^{(6)} = \frac{1225}{2579890176 q_0^{12}}$ and $\frac{d}{dt}\tau_I^{(2g)} = -\frac{d}{dt}F_I^{(2g)}$.

- Painlevé II:

$$\begin{aligned} F_{II}^{(4)} &= \frac{1}{480} \frac{(2048 q_0^{12} + 2560 \theta q_0^9 + 1280 \theta^2 q_0^6 + 1020 \theta^3 q_0^3 - 45 \theta^4) q_0^3}{\theta^2 (4 q_0^3 + \theta)^5} \\ F_{II}^{(6)} &= -\frac{q_0^6}{4032 \theta^4 (\theta + 4 q_0^3)^{10}} \left(4194304 q_0^{24} + 10485760 \theta q_0^{21} + 11796480 \theta^2 q_0^{18} \right) \end{aligned}$$

$$+7864320 \theta^3 q_0^{15} + 3440640 \theta^4 q_0^{12} - 5694528 \theta^5 q_0^9 + 5232752 \theta^6 q_0^6 - 510412 \theta^7 q_0^3 + 3969 \theta^8) \quad (I.21)$$

We can verify $(\dot{q}_0 = -\frac{q_0^2}{4q_0^3 + \theta})$ that $\frac{d}{dt} \tau_{\text{II}}^{(2g)} = -\frac{d}{dt} F_{\text{II}}^{(2g)}$ for $g = 2$ and $g = 3$.

- Painlevé III:

$$F_{\text{III}}^{(4)} = \frac{Q_{30}(q_0)}{240(\theta_0^2 - \theta_\infty^2)(-\theta_0 q_0^6 + 3\theta_\infty q_0^4 - 3\theta_0 q_0^2 + \theta_\infty)^5} \quad (I.22)$$

where $Q_{30}(q_0)$ is a even polynomial in q_0 of degree 30:

$$\begin{aligned} Q_{30}(q_0) = & \theta_\infty^3(10\theta_\infty^2\theta_0^2 + 7\theta_0^4 - \theta_0^4) + (-15\theta_\infty^2\theta_0(10\theta_\infty^2\theta_0^2 + 7\theta_0^4 - \theta_0^4))q_0^2 + 15\theta_\infty(-8\theta_0^6 + 46\theta_0^4\theta_\infty^2 + 9\theta_\infty^6 + 65\theta_\infty^2\theta_0^4)q_0^4 \\ & + (-5\theta_\infty^2\theta_0(205\theta_\infty^4 + 341\theta_0^4 + 910\theta_\infty^2\theta_0^2))q_0^6 + (-15\theta_\infty(-195\theta_0^6 - 652\theta_\infty^2\theta_0^4 - 631\theta_0^4\theta_\infty^2 + 22\theta_\infty^6))q_0^8 \\ & + (-3\theta_0(5212\theta_\infty^2\theta_0^4 + 8837\theta_\infty^4\theta_0^2 + 1494\theta_\infty^6 + 473\theta_0^6))q_0^{10} + 5\theta_\infty(534\theta_\infty^6 + 3893\theta_\infty^4\theta_0^2 + 9884\theta_\infty^2\theta_0^4 + 1705\theta_0^6)q_0^{12} \\ & + (-15\theta_0(13\theta_0^6 + 3465\theta_\infty^4\theta_0^2 + 2604\theta_\infty^2\theta_0^4 + 782\theta_\infty^6))q_0^{14} + 15\theta_\infty(3465\theta_\infty^2\theta_0^4 + 782\theta_\infty^6 + 13\theta_0^6 + 2604\theta_\infty^4\theta_0^2)q_0^{16} \\ & + (-5\theta_0(1705\theta_\infty^6 + 534\theta_0^6 + 3893\theta_\infty^2\theta_0^4 + 9884\theta_\infty^4\theta_0^2))q_0^{18} + 3\theta_\infty(1494\theta_0^6 + 473\theta_\infty^6 + 5212\theta_\infty^4\theta_0^2 + 8837\theta_\infty^2\theta_0^4)q_0^{20} \\ & + (-15\theta_0(631\theta_\infty^2\theta_0^4 + 652\theta_\infty^4\theta_0^2 - 22\theta_0^6 + 195\theta_\infty^6))q_0^{22} + 5\theta_\infty\theta_0^2(205\theta_0^4 + 910\theta_\infty^2\theta_0^2 + 341\theta_\infty^4)q_0^{24} \\ & + 15\theta_0(8\theta_\infty^6 - 65\theta_\infty^4\theta_0^2 - 9\theta_0^6 - 46\theta_\infty^2\theta_0^4)q_0^{26} + (-15\theta_\infty\theta_0^2(-10\theta_\infty^2\theta_0^2 + \theta_\infty^4 - 7\theta_0^4))q_0^{28} \\ & + \theta_0^3(-10\theta_\infty^2\theta_0^2 + \theta_\infty^4 - 7\theta_0^4)q_0^{30} \end{aligned} \quad (I.23)$$

One can verify $(\dot{q}_0 = \frac{(q_0^4 - 1)^2}{\theta_0 q_0^6 - 3\theta_\infty q_0^4 + 3\theta_0 q_0^2 - \theta_\infty})$ that $-\frac{d}{dt} F_{\text{III}}^{(4)} = -\frac{d}{dt} \tau_{\text{III}}^{(4)}$.

- Painlevé IV:

$$F_{\text{IV}}^{(4)} = -\frac{q_0^4 Q_9(t, q_0)}{960\theta_0^2(3q_0^4 + 2tq_0^3 + \theta_0^2)^5((tq_0 + q_0^2)^2 - \theta_0^2)^2} \quad (I.24)$$

with:

$$\begin{aligned} Q_9(t, q_0) = & 243q_0^{24} - 603q_0^{20}\theta_0^2 + 353q_0^4\theta_0^{10} - 16\theta_0^{12} - 3474q_0^{16}\theta_0^4 + 1962q_0^{12}\theta_0^6 - 2561q_0^8\theta_0^8 \\ & + q_0^3(1782q_0^{20} - 1593q_0^{16}\theta_0^2 + 8406q_0^8\theta_0^6 - 4762q_0^4\theta_0^8 + 91\theta_0^{10} - 16212q_0^{12}\theta_0^4)t \\ & + 2q_0^6(6690q_0^4\theta_0^6 + 582q_0^{12}\theta_0^2 - 1525\theta_0^8 + 2889q_0^{16} - 16188q_0^8\theta_0^4)t^2 \\ & - q_0^5(589\theta_0^8 - 9569q_0^{12}\theta_0^2 - 10872q_0^{16} - 10299q_0^4\theta_0^6 + 35655q_0^8\theta_0^4)t^3 \\ & + 3q_0^8(1289\theta_0^6 + 5303q_0^8\theta_0^2 + 4361q_0^{12} - 7785q_0^4\theta_0^4)t^4 \\ & + q_0^7(10442q_0^{12} - 9120q_0^4\theta_0^4 + 545\theta_0^6 + 13365q_0^8\theta_0^2)t^5 \\ & + 4q_0^{10}(1382q_0^8 + 1573q_0^4\theta_0^2 - 491\theta_0^4)t^6 - q_0^9(175\theta_0^4 - 1591q_0^4\theta_0^2 - 1872q_0^8)t^7 \\ & + 2q_0^{12}(184q_0^4 + 85\theta_0^2)t^8 + 32q_0^{15}t^9 \end{aligned} \quad (I.25)$$

One can verify $(\dot{q}_0 = -\frac{q_0^3(t+2q_0)}{3q_0^4+2tq_0^3+\theta_0^2})$ that $-\frac{d}{dt} F_{\text{IV}}^{(4)} = \frac{d}{dt} \tau_{\text{IV}}^{(4)}$.

I.4 Details for the computation $F_{\text{VI}}^{(0)}$

I.4.1 Spectral curve

The spectral curve for (P_{VI}) is given by

$$y^2 = \frac{\theta_\infty^2(x - q_0)^2 P(x)}{4x^2(x - 1)^2(x - t)^2} \quad \text{with } P(x) = x^2 + \left(-1 - \frac{\tilde{\theta}_0^2 t^2}{q_0^2} + \frac{\tilde{\theta}_1^2(t - 1)^2}{(q_0 - 1)^2}\right)x + \frac{\tilde{\theta}_0^2 t^2}{q_0^2}, \quad (I.26)$$

where we remind the reader that the reduced monodromy parameters are $\tilde{\theta}_i = \frac{\theta_i}{\theta_\infty}$. Note that $P(x)$ also admits a more symmetric formulation. Using the algebraic equation satisfied by q_0 , we can reformulate:

$$P(x) = x^2 + \left(-\frac{\tilde{\theta}_0^2 t(t + 1)}{q_0^2} + \frac{\tilde{\theta}_1^2 t(t - 1)}{(q_0 - 1)^2} - \frac{\tilde{\theta}_t^2 t(t - 1)}{(q_0 - t)^2}\right)x + \frac{\tilde{\theta}_0^2 t^2}{q_0^2}. \quad (I.27)$$

Equivalently, it is also defined by the following 3 conditions:

$$P(0) = \frac{\tilde{\theta}_0^2 t^2}{q_0^2}, \quad P(1) = \frac{(t-1)^2 \tilde{\theta}_1^2}{(q_0-1)^2} \text{ and } P(t) = \frac{t^2(t-1)^2 \tilde{\theta}_t^2}{(q_0-t)^2}. \quad (\text{I.28})$$

Since the spectral curve is of genus 0, we can parametrize it globally. Let us define the following parametrizations:

$$\begin{aligned} x(z) &= \frac{a+b}{2} + \frac{b-a}{4} \left(z + \frac{1}{z} \right) \text{ where by definition } P(x) = (x-a)(x-b) \\ x(z) &= 0 + \frac{b-a}{4z} (z - z_{0,+})(z - z_{0,-}) \text{ where by definition } x(z_{0,+}) = x(z_{0,-}) = 0 \\ x(z) &= 1 + \frac{b-a}{4z} (z - z_{1,+})(z - z_{1,-}) \text{ where by definition } x(z_{1,+}) = x(z_{1,-}) = 1 \\ x(z) &= t + \frac{b-a}{4z} (z - z_{t,+})(z - z_{t,-}) \text{ where by definition } x(z_{t,+}) = x(z_{t,-}) = t \\ x(z) &= q_0 + \frac{b-a}{4z} (z - z_{q_0,+})(z - z_{q_0,-}) \text{ where by definition } x(z_{q_0,+}) = x(z_{q_0,-}) = q_0 \end{aligned} \quad (\text{I.29})$$

Thus the branchpoints are located at $z = \pm 1$ and we can rewrite the spectral curve and the 1-form $w = ydx$:

$$y(z) = \frac{2\theta_\infty(z - z_{q_0,+})(z - z_{q_0,-})(z^2 - 1)z}{(b-a)(z - z_{0,+})(z - z_{0,-})(z - z_{1,+})(z - z_{1,-})(z - z_{t,+})(z - z_{t,-})}, \quad (\text{I.30})$$

$$\begin{aligned} w(z) &= y(z)dx(z) \\ &= \frac{\theta_\infty(z - z_{q_0,+})(z - z_{q_0,-})(z^2 - 1)^2}{2z(z - z_{0,+})(z - z_{0,-})(z - z_{1,+})(z - z_{1,-})(z - z_{t,+})(z - z_{t,-})} dz. \end{aligned} \quad (\text{I.31})$$

The important point is to notice that $w(z)$ is a meromorphic 1-form with simple poles only at $z \in \{0, z_{i,\pm}, \infty\}$ with $i \in \{0, 1, t\}$. Analyzing the different residues at these points and using (D.12) we get that:

$$w(z) = \left(\frac{\theta_\infty}{2z} - \frac{\theta_0}{2(z - z_{0,+})} + \frac{\theta_0}{2(z - z_{0,-})} + \frac{\theta_1}{2(z - z_{1,+})} - \frac{\theta_1}{2(z - z_{1,-})} - \frac{\theta_t}{2(z - z_{t,+})} + \frac{\theta_t}{2(z - z_{t,-})} \right) dz. \quad (\text{I.32})$$

Note that in particular the ambiguity between $z_{i,+}$ and $z_{i,-}$ is settled by the choice of the sign in the previous residues. We adopted for convenience a sign difference for θ_1 . In fact, one can verify that equations (I.30) and (I.32) are equivalent to the algebraic equation for $q_0(t)$. Let us now try to compute the first symplectic invariant $F^{(0)}$ for our spectral curve.

I.4.2 Computation of $F_{\text{VI}}^{(0)}$

Following the Eynard-Orantin framework [16], we observe that $w(z)$ has 8 singular points: $z \in \{0, \infty, z_{0,+}, z_{0,-}, z_{1,+}, z_{1,-}, z_{t,+}, z_{t,-}\}$. In the Eynard-Orantin language, $z = 0$ and $z = \infty$ are type 1 singular points since they are simple poles of $x(z)$ (but not of $y(z)$). On the other hand, the remaining six singular points fall into the type 2 category, i.e. are poles of $y(z)$. Hence the local coordinates are:

$$z_\alpha(p) = \begin{cases} x(p) & \text{for } z \in \{0, \infty\}, \\ \frac{1}{x(p) - x(\alpha)} & \text{for } z \in \{z_{0,\pm}, z_{1,\pm}, z_{t,\pm}\}. \end{cases}$$

It is easy to see that there is no potential $V_\alpha(p)$ for any points. The temperatures are given from (I.32):

α	t_α
0	$\frac{\theta_\infty}{2}$
∞	$-\frac{\theta_\infty}{2}$
$z_{0,+}$	$-\frac{\theta_0}{2}$
$z_{0,-}$	$\frac{\theta_0}{2}$
$z_{1,+}$	$\frac{\theta_1}{2}$
$z_{1,-}$	$-\frac{\theta_1}{2}$
$z_{t,+}$	$-\frac{\theta_t}{2}$
$z_{t,-}$	$\frac{\theta_t}{2}$

In order to compute the μ_α 's we observe that:

$$\begin{aligned}
\forall i \in \{0, 1, t\} : \frac{dz_{\alpha_i}(p)}{z_{\alpha_i}(p)} &= \frac{dp}{p} - \frac{dp}{p - z_{i,+}} - \frac{dp}{p - z_{i,-}} \\
\text{For } z = 0 : \frac{dz_\alpha(p)}{z_\alpha(p)} &= -\frac{1}{p} + d_p \left(\ln(1 + p^2 + \frac{2(a+b)p}{b-a}) \right) \\
\text{For } z = \infty : \frac{dz_\alpha(p)}{z_\alpha(p)} &= \frac{1}{p} + d_p \left(\ln(1 + \frac{1}{p^2} + \frac{2(a+b)}{(b-a)p}) \right)
\end{aligned} \tag{I.33}$$

Then a tedious computation from Eynard-Orantin formula ([16, §4.2.2]) shows that we obtain:

$$\begin{aligned}
F_{\text{VI}}^{(0)} &= \frac{\theta_0^2 + \theta_1^2 - \theta_\infty^2 + \theta_t^2}{8} \ln \frac{(b-a)^2}{16} \\
&+ \frac{1}{8} \left[\theta_0^2 \ln \frac{(z_{0,+} - z_{0,-})^4}{z_{0,+}z_{0,-}} + \theta_1^2 \ln \frac{(z_{1,+} - z_{1,-})^4}{z_{1,+}z_{1,-}} + \theta_t^2 \ln \frac{(z_{t,+} - z_{t,-})^4}{z_{t,+}z_{t,-}} \right] \\
&+ \frac{1}{4} \left[\theta_0 \theta_\infty \ln \frac{z_{0,+}}{z_{0,-}} - \theta_1 \theta_\infty \ln \frac{z_{1,+}}{z_{1,-}} + \theta_t \theta_\infty \ln \frac{z_{t,+}}{z_{t,-}} \right] \\
&+ \frac{1}{4} \left[\theta_0 \theta_1 \ln \frac{(z_{1,+} - z_{0,+})(z_{1,-} - z_{0,-})}{(z_{1,+} - z_{0,-})(z_{0,+} - z_{1,-})} + \theta_1 \theta_t \ln \frac{(z_{1,+} - z_{t,+})(z_{1,-} - z_{t,-})}{(z_{1,+} - z_{t,-})(z_{t,+} - z_{1,-})} \right. \\
&\left. - \theta_0 \theta_t \ln \frac{(z_{t,+} - z_{0,+})(z_{t,-} - z_{0,-})}{(z_{t,+} - z_{0,-})(z_{0,+} - z_{t,-})} \right]
\end{aligned} \tag{I.34}$$

Since $\forall i \in \{0, 1, t\} : z_{i,+}z_{i,-} = 1$ the expression for $F_{\text{VI}}^{(0)}$ simplifies into:

$$\begin{aligned}
F_{\text{VI}}^{(0)} &= \frac{\theta_0^2 + \theta_1^2 - \theta_\infty^2 + \theta_t^2}{8} \ln \frac{(b-a)^2}{16} \\
&+ \frac{1}{8} \left[\theta_0^2 \ln \frac{(z_{0,+}^2 - 1)^4}{z_{0,+}^4} + \theta_1^2 \ln \frac{(z_{1,+}^2 - 1)^4}{z_{1,+}^4} + \theta_t^2 \ln \frac{(z_{t,+}^2 - 1)^4}{z_{t,+}^4} \right] \\
&+ \frac{1}{4} \left[\theta_0 \theta_\infty \ln z_{0,+}^2 - \theta_1 \theta_\infty \ln z_{1,+}^2 + \theta_t \theta_\infty \ln z_{t,+}^2 \right] \\
&+ \frac{1}{4} \left[\theta_0 \theta_1 \ln \left(-\frac{(z_{1,+} - z_{0,+})^2}{(1 - z_{1,+}z_{0,+})^2} \right) + \theta_1 \theta_t \ln \left(-\frac{(z_{1,+} - z_{t,+})^2}{(1 - z_{1,+}z_{t,+})^2} \right) \right. \\
&\left. - \theta_0 \theta_t \ln \left(-\frac{(z_{t,+} - z_{0,+})^2}{(1 - z_{t,+}z_{0,+})^2} \right) \right]
\end{aligned} \tag{I.35}$$

Ingenious computations shows that we have:

$$\begin{aligned}
ab &= \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}, \\
a+b &= 1 + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2} - \frac{\theta_1^2 (t-1)^2}{\theta_\infty^2 (q_0-1)^2} = t \left[1 + \frac{\theta_0^2}{\theta_\infty^2 q_0^2} - \frac{\theta_t^2 (t-1)^2}{\theta_\infty^2 (q_0-t)^2} \right] \\
&= t + 1 + \frac{(t-1)\theta_1^2}{(q_0-1)^2 \theta_\infty^2} - \frac{t^2(t-1)\theta_t^2}{(q_0-t)^2 \theta_\infty^2}
\end{aligned} \tag{I.36}$$

Consequently using $(b-a)^2 = (a+b)^2 - 4ab$ we find three equivalent expressions for $(b-a)^2$:

$$\begin{aligned}
(b-a)^2 &= \prod_{\pm} \left(1 \pm \frac{\theta_0 t}{\theta_\infty q_0} \pm \frac{\theta_1 (t-1)}{\theta_\infty (q_0-1)} \right) \\
&= t^2 \prod_{\pm} \left(1 \pm \frac{\theta_0}{\theta_\infty q_0} \pm \frac{\theta_t (t-1)}{\theta_\infty (q_0-t)} \right) \\
&= (t-1)^2 \prod_{\pm} \left(1 \pm \frac{\theta_1}{\theta_\infty (q_0-1)} \pm \frac{\theta_t t}{\theta_\infty (q_0-t)} \right).
\end{aligned} \tag{I.37}$$

Here each product is taken over the four possibilities for the signs. We know want to define non-ambiguously $z_{0,+}$, $z_{1,+}$ and $z_{t,+}$, from the definition (I.29). We have:

$$z_{i,+} + \frac{1}{z_{i,+}} = \frac{2(2i-a-b)}{b-a} \text{ for } i \in \{0, 1, t\}. \tag{I.38}$$

Moreover the points $z_{0,+}$, $z_{1,+}$ and $z_{t,+}$ are defined such that:

$$\text{Res}_{z \rightarrow z_{0,+}} w(z) dz = -\frac{\theta_0}{2}, \quad \text{Res}_{z \rightarrow z_{1,+}} w(z) dz = \frac{\theta_1}{2}, \quad \text{Res}_{z \rightarrow z_{t,+}} w(z) dz = -\frac{\theta_t}{2}, \tag{I.39}$$

where:

$$w(z) = \frac{\theta_\infty (x(z) - q_0) (b-a)^2 (z^2 - 1)^2}{32 z^3 x(z) (x(z) - 1) (x(z) - t)}. \tag{I.40}$$

Computing the various residues gives that:

$$\begin{aligned}
z_{0,+} - \frac{1}{z_{0,+}} &= \frac{4\theta_0 t}{\theta_\infty q_0 (b-a)} \\
z_{1,+} - \frac{1}{z_{1,+}} &= \frac{4\theta_1 (t-1)}{\theta_\infty (q_0-1) (b-a)} \\
z_{t,+} - \frac{1}{z_{t,+}} &= \frac{4\theta_t t (t-1)}{\theta_\infty (q_0-t) (b-a)}.
\end{aligned} \tag{I.41}$$

Combining both (I.38) with (I.41) gives:

$$\begin{aligned}
z_{0,+} &= \frac{1}{b-a} \left(\frac{2\theta_0 t}{\theta_\infty q_0} - a - b \right) \\
z_{1,+} &= \frac{1}{b-a} \left(\frac{2\theta_1 (t-1)}{\theta_\infty (q_0-1)} + 2 - a - b \right) \\
z_{t,+} &= \frac{1}{b-a} \left(\frac{2\theta_t t (t-1)}{\theta_\infty (q_0-t)} + 2t - a - b \right)
\end{aligned} \tag{I.42}$$

We can now replace $a + b$ using expression given in (I.36). We find the following values for $z_{i,+}$:

$$\begin{aligned}
z_{0,+} &= -\frac{1}{b-a} \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \\
z_{0,+} &= -\frac{t}{b-a} \left(1 - \frac{\theta_0}{\theta_\infty q_0} - \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \left(1 - \frac{\theta_0}{\theta_\infty q_0} + \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \\
z_{1,+} &= \frac{1}{b-a} \left(1 + \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \\
z_{1,+} &= -\frac{t-1}{b-a} \left(1 - \frac{\theta_1}{\theta_\infty(q_0-1)} + \frac{t\theta_t}{\theta_\infty(q_0-t)} \right) \left(1 - \frac{\theta_1}{\theta_\infty(q_0-1)} - \frac{t\theta_t}{\theta_\infty(q_0-t)} \right) \\
z_{t,+} &= \frac{t}{b-a} \left(1 + \frac{\theta_0}{\theta_\infty q_0} + \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \left(1 - \frac{\theta_0}{\theta_\infty q_0} + \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \\
z_{t,+} &= \frac{t-1}{b-a} \left(1 + \frac{\theta_1}{\theta_\infty(q_0-1)} + \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \left(1 - \frac{\theta_1}{\theta_\infty(q_0-1)} + \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \quad (\text{I.43})
\end{aligned}$$

We now regroup these results for the computation of $F^{(0)}$ given by (I.35). We first observe that the terms involving θ_i^2 's are given by:

Term for θ_0^2	$\frac{1}{8} \ln \left(\frac{16 \theta_0^4 t^4}{\theta_\infty^4 q_0^4 \prod_{\pm} \left(1 \pm \frac{\theta_0 t}{\theta_\infty q_0} \pm \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)} \right)$
Term for θ_1^2	$\frac{1}{8} \ln \left(\frac{16 \theta_1^4 (t-1)^4}{\theta_\infty^4 (q_0-1)^4 \prod_{\pm} \left(1 \pm \frac{\theta_0 t}{\theta_\infty q_0} \pm \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)} \right)$
Term for θ_t^2	$\frac{1}{8} \ln \left(\frac{16 \theta_t^4 t^4 (t-1)^4}{\theta_\infty^4 (q_0-t)^4 \prod_{\pm} \left(1 \pm \frac{\theta_0 t}{\theta_\infty q_0} \pm \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)} \right)$
Term for θ_∞^2	$-\frac{1}{8} \ln \left(\frac{1}{16} \prod_{\pm} \left(1 \pm \frac{\theta_0 t}{\theta_\infty q_0} \pm \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \right)$

Additionally we can obtain the cross-terms (we looked for the expression minimizing the appearance of θ_t in order to match it more easily with (I.10), and to obtain a symmetric expression in $(\theta_0, \theta_1, \theta_t)$, one can easily take each contribution symmetrical relatively to these variables using (I.43)):

Term for $\theta_0 \theta_\infty$	$\frac{1}{4} \ln \left(\frac{\left(1 - \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)}{\left(1 + \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 + \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)} \right)$
Term for $\theta_1 \theta_\infty$	$\frac{1}{4} \ln \left(\frac{\left(1 - \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 + \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)}{\left(1 + \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)} \right)$
Term for $\theta_t \theta_\infty$	$\frac{1}{4} \ln \left(\frac{\left(1 + \frac{\theta_0}{\theta_\infty q_0} + \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \left(1 - \frac{\theta_0}{\theta_\infty q_0} + \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right)}{\left(1 - \frac{\theta_0}{\theta_\infty q_0} - \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right) \left(1 + \frac{\theta_0}{\theta_\infty q_0} - \frac{\theta_t(t-1)}{\theta_\infty(q_0-t)} \right)} \right)$
Term for $\theta_0 \theta_1$	$\frac{1}{4} \ln \left(-\frac{\left(1 + \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)}{\left(1 + \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)} \right)$
Term for $\theta_0 \theta_t$	$\frac{1}{4} \ln \left(-\frac{\left(1 + \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)}{\left(1 + \frac{\theta_0 t}{\theta_\infty q_0} - \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right) \left(1 - \frac{\theta_0 t}{\theta_\infty q_0} + \frac{\theta_1(t-1)}{\theta_\infty(q_0-1)} \right)} \right)$
Term for $\theta_1 \theta_t$	$\frac{1}{4} \ln \left(-\frac{\left(1 + \frac{\theta_1}{\theta_\infty(q_0-1)} - \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \left(1 - \frac{\theta_1}{\theta_\infty(q_0-1)} + \frac{\theta_t t}{\theta_\infty(q_0-t)} \right)}{\left(1 + \frac{\theta_1}{\theta_\infty(q_0-1)} + \frac{\theta_t t}{\theta_\infty(q_0-t)} \right) \left(1 - \frac{\theta_1}{\theta_\infty(q_0-1)} - \frac{\theta_t t}{\theta_\infty(q_0-t)} \right)} \right)$

We now have all the ingredients to compute explicitly $F_{\text{VI}}^{(0)}$ from (I.35) and we find (I.9).

It is then easy to verify that:

$$\frac{d}{dt}F_{\text{VI}}^{(0)}(t, q_0) = \dot{q}_0 \frac{\partial}{\partial q_0} F_{\text{VI}}^{(0)}(t, q_0) + \frac{\partial}{\partial t} F_{\text{VI}}^{(0)}(t, q_0) = -\frac{d}{dt}\dot{\tau}_0 \quad (\text{I.44})$$

Remark I.1 *Computation of $F_{\text{VI}}^{(2)}$ follows the general topological recursion as presented in [16] (see Definition 5.4). Computations rapidly become impossible to handle with a standard laptop to simplify expressions. However we could verify explicitly that the following identity holds:*

$$\frac{d}{dt}F_{\text{VI}}^{(2)}(t, q_0) = \frac{\partial}{\partial t} F_{\text{VI}}^{(2)}(t, q_0) + \dot{q}_0 \frac{\partial}{\partial q_0} F_{\text{VI}}^{(2)}(t, q_0) = -\frac{d}{dt}\tau_{\text{VI}}^{(4)} \quad (\text{I.45})$$

Unfortunately our final expression for $F_{\text{VI}}^{(2)}(t, q_0)$ is several pages long and presents no particular interest but for the fact that its derivative recovers $-\frac{d}{dt}\tau_{\text{VI}}^{(4)}$. Hence we do not reproduce it here.